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q-Deformed quantum Lie algebras

Alexander Schmidt[*,1], Hartmut Wachter[1]

Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany

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Abstract

Attention is focused on q-deformed quantum algebras with physical importance, i.e. $U_q(su_2)$, $U_q(so_4)$ and q-deformed Lorentz algebra. The main concern of this article is to assemble important ideas about these symmetry algebras in a consistent framework which will serve as starting point for representation theoretic investigations in physics, especially quantum field theory. In each case considerations start from a realization of symmetry generators within the differential algebra. Formulae for coproducts and antipodes on symmetry generators are listed. The action of symmetry generators in terms of their Hopf structure is taken as the q-analog of classical commutators and written out explicitly. Spinor and vector representations of symmetry generators are calculated. A review of the commutation relations between symmetry generators and components of a spinor or vector operator is given. Relations for the corresponding quantum Lie algebras are computed. Their Casimir operators are written down in a form similar to that for the undeformed case.

Keywords: Quantum groups; q-Deformation; Quantum Lie algebras; Non-commutative geometry; Spacetime symmetries

1. Introduction

It is an old idea that limiting the precision of position measurements by a fundamental length will lead to a new method for regularizing quantum field theories [20]. It is also well known that

* Corresponding address: Ludwig-Maximilians-Universitaet, Department of Physics, Arnold-Sommerfeld-Center, Theresienstr. 37, D-80333 Munich, Germany. Tel.: +49 89 2180 4379.
E-mail addresses: schmidt@theorie.physik.uni-muenchen.de (A. Schmidt), Hartmut.Wachter@physik.uni-muenchen.de (H. Wachter).
1 Present address: Arnold-Sommerfeld-Center, Ludwig-Maximilians-Universität, Theresienstr. 37, D-80333 München, Germany.

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such a modification of classical spacetime will in general break its Poincaré symmetry [45]. One way out of this difficulty is to change not only spacetime, but also its underlying symmetry.

Quantum groups can be seen as deformations of classical spacetime symmetries, as they describe the symmetry of their comodules, the so-called quantum spaces. From a physical point of view the most realistic examples for quantum groups and quantum spaces arise from q-deformation [14,15,21,24,33,39,54]. In our work we are interested in q-deformed versions of Minkowski space and Euclidean spaces as well as their corresponding symmetries, given by q-deformed Lorentz algebra and algebras of q-deformed angular momentum, respectively [6,26,29,38,42]. Remarkably, Julius Wess and his coworkers were able to show that q-deformation of spaces and symmetries can indeed lead to discretizations, as they result from the existence of a smallest distance [9,16]. This observation nourishes the hope that q-deformation might give a new method for regularizing quantum field theories [5,19,28,36].

In our previous work [1,34,43,48–51] attention was focused on q-deformed quantum spaces of physical importance, i.e. the two-dimensional Manin plane, q-deformed Euclidean space in three or four dimensions and q-deformed Minkowski space. If we want to describe fields on q-deformed quantum spaces we need to consider representations of the corresponding quantum symmetries, given by $U_q(su_2)$, $U_q(so_4)$ and q-deformed Lorentz algebra. The study of such quantum algebras has produced a number of remarkable results during the last two decades. For a review we recommend to the reader the presentations in [11,22,32] and references therein. In this article we want to adapt these general ideas to our previous considerations about q-deformed quantum spaces. In doing so, we provide a basis for performing concrete calculations, as they are necessary in formulating and evaluating field theories on quantum spaces.

In particular, we intend to proceed as follows. In Section 2 we cover the ideas that our considerations about q-deformed quantum symmetries are based on. In the subsequent sections we first recall for each quantum algebra under consideration how its generators are realized within the corresponding q-deformed differential calculus. Then we are going to present explicit formulae for coproduct and antipode on a set of independent symmetry generators. With this knowledge at hand we should be able to write down explicit formulae for so-called q-commutators between symmetry generators and representation space elements. In addition to this, we are going to consider spinor and vector representations of the independent symmetry generators and give a complete review of the commutation relations between symmetry generators and components of a spinor or vector operator. Furthermore we are going to calculate the adjoint action of the independent symmetry generators on each other. In this manner, we will get relations for quantum Lie algebras. We will close our considerations by writing down q-analogs of Casimir operators. Finally, Section 6 will serve as a short conclusion. For reference and for the purpose of introducing consistent and convenient notation, we provide a review of key notation and results in Appendix A.

We should also mention that most of our results were obtained by applying the computer algebra system Mathematica [53]. We are convinced that in the future the use of this powerful tool will be inevitable in managing the extraordinary complexity of q-deformation.

2. Basic ideas on q-deformed quantum symmetries

Roughly speaking a quantum space is nothing else than an algebra generated by non-commuting coordinates $X_1, X_2, \ldots, X_n$, i.e.

$$\mathcal{A}_q = \mathbb{C}[[X_1, \ldots, X_n]]/\mathcal{I}, \tag{1}$$

where $\mathcal{I}$ denotes the ideal generated by the relations of the non-commuting coordinates. The quantum spaces we are interested in for physical reasons are the two-dimensional Manin plane, q-deformed Euclidean space in three or four dimensions as well as q-deformed Minkowski space. For their definition we refer the reader to Appendix A.

On each of these quantum spaces there exist two differential calculi [7,46,52] with

$$\partial^i X^j = g^{ij} + k(\hat{R}^{-1})^{ij}_{kl} X^k \partial^l, \quad k \in \mathbb{R},$$
$$\hat{\partial}^i X^j = g^{ij} + k^{-1}(\hat{R})^{ij}_{kl} X^k \hat{\partial}^l, \tag{2}$$

where $\hat{R}$ and g^{ij} denote respectively the R-matrix and the corresponding quantum metric of the underlying quantum symmetry. There is a q-deformed antisymmetrizer P_A that enables us to define the components of q-deformed orbital angular momentum as antisymmetrized products of coordinates and derivatives [18]. Specifically, we have

$$L^{ij} \sim (P_A)^{ij}_{kl} X^k \partial^l \sim (P_A)^{ij}_{kl} X^k \hat{\partial}^l. \tag{3}$$

It can be shown that the L^{ij} together with a central generator U subject to

$$U - 1 \sim \lambda g_{ij} X^i \hat{\partial}^j, \quad \lambda \equiv q - q^{-1}, \tag{4}$$

span the quantum algebras describing the underlying quantum symmetry. In this way we will obtain $U_q(su_2)$, $U_q(so_4)$ as well as q-deformed Lorentz algebra in the remainder of this article.

In accordance with the classical case, a single particle wave function should be defined as an element of the tensor product of a finite vector space $\mathcal{S}$ holding the spin degrees of freedom and the algebra of space functions $\mathcal{M}$, where $\mathcal{S}$ and $\mathcal{M}$ are both modules of the q-deformed symmetry algebra under consideration. In contrast to the classical situation, we now have to distinguish between right and left wave functions, i.e.

$$\psi_R \equiv \sum_j e^j \otimes \psi^j, \quad \psi_L \equiv \sum_j \psi^j \otimes e^j, \tag{5}$$

where $\{e^j\}$ is a basis in $\mathcal{S}$ and ψ^j are elements of $\mathcal{M}$. The reason for this lies in the fact that due to the braiding between the two tensor factors we are not allowed to interchange them by a trivial flip [30].

The transformation properties of our wave functions are determined by the coproduct of the symmetry generators, since we have (we write the coproduct in the so-called Sweedler notation, i.e. $\Delta(a) = a_{(1)} \otimes a_{(2)}$)

$$L^{ij} \triangleright \psi_R = \sum_k (L^{ij})_{(1)} \triangleright e^k \otimes (L^{ij})_{(2)} \triangleright \psi^k, \tag{6}$$
$$L^{ij} \triangleright \psi_L = \sum_k (L^{ij})_{(1)} \triangleright \psi^k \otimes (L^{ij})_{(2)} \triangleright e^k.$$

For scalar wave functions $\mathcal{S}$ has to be a one-dimensional vector space with the corresponding representation on it being the trivial one. In this manner, the symmetry transformation is reduced to the action on space functions. In Ref. [1] we derived explicit formulae for such actions. In the rest of this article we will restrict ourselves to representations of symmetry generators on a spinorial or vectorial basis. Physical fields of higher spin should then be built up from spinor or vector fields.

It should be noted that we can combine a quantum algebra $\mathcal{H}$ with its representation space $\mathcal{A}$ to form a left cross-product algebra $\mathcal{A} \rtimes \mathcal{H}$ built on $\mathcal{A} \otimes \mathcal{H}$ with product

$$(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b) \otimes h_{(2)}g, \quad a, b \in \mathcal{A}, h, g \in \mathcal{H}. \tag{7}$$

There is also a right-handed version of this notion called a right cross-product algebra $\mathcal{H} \ltimes \mathcal{A}$ and built on $\mathcal{H} \otimes \mathcal{A}$ with product

$$(h \otimes a)(g \otimes b) = hg_{(1)} \otimes (a \triangleleft g_{(2)})b. \tag{8}$$

The last two identities tell us that the commutation relations between symmetry generators and representation space elements are completely determined by the coproduct and action of the symmetry generators.

In this article it can be seen that there is a remarkable correspondence between q-deformed symmetry algebras and their classical counterparts. Towards this end we have to introduce the notion of a q-commutator which is nothing other than the action of a symmetry generator L^{ij} on a representation space element V. Expressing that action in terms of the Hopf structure of L^{ij} the q-brackets become

$$\begin{aligned} [L^{ij}, V]_q &\equiv L^{ij} \triangleright V = L^{ij}_{(1)} V S(L^{ij}_{(2)}), \\ [V, L^{ij}]_q &\equiv V \triangleleft L^{ij} = S^{-1}(L^{ij}_{(2)}) V L^{ij}_{(1)}, \end{aligned} \tag{9}$$

where S and S^{-1} denote the antipode and its inverse, respectively. From their very definition it follows that q-commutators obey the q-deformed Jacobi identities

$$\begin{aligned} [L^{ij}, [L^{kl}, V]_q]_q &= [[L^{ij}_{(1)}, L^{kl}]_q, [L^{ij}_{(2)}, V]_q]_q, \\ [[L^{ij}, L^{kl}]_q, V]_q &= [L^{ij}_{(1)}, [L^{kl}, [S(L^{ij}_{(2)}), V]_q]_q]_q. \end{aligned} \tag{10}$$

Now, we are able to introduce the notion of a quantum Lie algebra as it was given in [31,44,47]. A quantum Lie algebra can be regarded as a subspace of a q-deformed enveloping algebra $U_q(g)$ being invariant under the adjoint action of $U_q(g)$. The point now is that the L^{ij} are the components of a tensor operator and this is their adjoint actions on each other equal a linear combination of the L^{ij} [3], i.e. the L^{ij} span a quantum Lie algebra with

$$[L^{ij}, L^{kl}]_q (= L^{ij} \triangleright L^{kl} = L^{ij} \triangleleft L^{kl}) = (C^{ij})^{kl}_{mn} L^{mn}, \tag{11}$$

where the C^{ij} are the so-called quantum structure constants. In the subsequent sections we are going to determine those constants in the case of $U_q(su_2)$, $U_q(so_4)$ and q-deformed Lorentz algebra.

Finally, let us mention that the q-brackets give another way to write down commutation relations between symmetry generators and components of a vector or spinor operator. More formally, we have

$$[L^{ij}, X^k]_q = (\tau^{ij}_L)^k{}_l X^l, \quad [X_k, L^{ij}]_q = X_l\, (\tau^{ij}_R)^l{}_k, \tag{12}$$

and

$$[L^{ij}, \theta^\alpha]_q = (\sigma^{ij}_L)^\alpha{}_\beta \theta^\beta, \quad \left[\theta^\alpha, L^{ij}\right]_q = \theta_\beta\, (\sigma^{ij}_R)^\beta{}_\alpha, \tag{13}$$

where X^k and θ^α stand respectively for components of vector and spinor operators. Notice that for $\tau_{L/R}$ and $\sigma_{L/R}$ we have to substitute the representation matrix of L^{ij}, as it comes out for the vector and spinor representations, respectively. It should be appreciated that the relations in (11)–(13) become part of a q-deformed super-Euclidean or super-Poincaré algebra, if such objects exist.

3. Quantum Lie algebra of three-dimensional angular momentum

3.1. Representation of three-dimensional angular momentum within q-deformed differential calculus

In the case of three-dimensional q-deformed Euclidean space (for its definition see Appendix A) the generators of orbital angular momentum are defined by [26]

$$L^A \equiv \Lambda^{1/2} X^C \hat{\partial}^D \epsilon_{DC}{}^A, \quad A \in \{+, 3, -\}, \tag{14}$$

where $\varepsilon_{DC}{}^A$ denotes a q-analog of the completely antisymmetric tensor of third rank and Λ stands for a scaling operator subject to

$$\Lambda X^A = q^4 X^A \Lambda, \quad \Lambda \hat{\partial}^A = q^{-4} \hat{\partial}^A \Lambda, \quad A \in \{+, 3, -\}. \tag{15}$$

If not stated otherwise summation over all repeated indices is to be understood. Substituting for $\varepsilon_{DC}{}^A$ their explicit form, we obtain from Eq. (14)

$$\begin{aligned} L^+ &= -q^{-1} \Lambda^{1/2} X^+ \hat{\partial}^3 + q^{-3} \Lambda^{1/2} X^3 \hat{\partial}^+, \\ L^3 &= -q^{-2} \Lambda^{1/2} X^+ \hat{\partial}^- + q^{-2} \Lambda^{1/2} X^- \hat{\partial}^+ - q^{-2} \lambda \Lambda^{1/2} X^3 \hat{\partial}^3, \\ L^- &= -q^{-1} \Lambda^{1/2} X^3 \hat{\partial}^- + q^{-3} \Lambda^{1/2} X^- \hat{\partial}^3. \end{aligned} \tag{16}$$

Using the Leibniz rules for partial derivatives in the form

$$X^A \hat{\partial}^B = g^{AB} + (\hat{R}^{-1})^{AB}{}_{CD} \hat{\partial}^C X^D, \tag{17}$$

and taking into account the identities

$$g^{AB} \epsilon_{BA}{}^C = 0, \quad (\hat{R}^{-1})^{AB}{}_{CD} \epsilon_{BA}{}^E = -q^4 \epsilon_{DC}{}^E, \tag{18}$$

the generators in Eq. (14) can alternatively be written as

$$L^A = -q^4 \Lambda^{1/2} \hat{\partial}^C X^D \epsilon_{DC}{}^A. \tag{19}$$

As already mentioned, there is a second set of derivatives ∂^A which can be linked to the first one via the relation [26]

$$\hat{\partial}^A = \Lambda^{-1} (\partial^A + q^3 \lambda X^A \partial^B \partial^C g_{BC}). \tag{20}$$

Making use of this identity together with $X^C X^D \epsilon_{DC}{}^A = 0$, one can show that we additionally have

$$L^A = q^4 \Lambda^{-1/2} X^C \partial^D \epsilon_{DC}{}^A = -\Lambda^{-1/2} \partial^C X^D \epsilon_{DC}{}^A. \tag{21}$$

3.2. Hopf structure of $U_q(su_2)$ and corresponding q-commutators

As is well known, the generators L^+, L^3, and L^- together with a generator $\tau^{1/2}$ can be viewed as elements of the quantum algebra $U_q(su_2)$ [26]. Its defining relations read

$$\begin{aligned}
&\tau^{1/2}L^{\pm} = q^{\mp 2}L^{\pm}\tau^{1/2}, \quad \tau^{1/2}L^3 = L^3\tau^{1/2}, \\
&L^-L^+ - L^+L^- = \tau^{-1/2}L^3, \\
&L^{\pm}L^3 - L^3L^{\pm} = \mp q^{\pm 1}L^{\pm}\tau^{-1/2}.
\end{aligned} \tag{22}$$

Furthermore, this algebra has a Hopf structure given by

$$\begin{aligned}
\Delta(L^{\pm}) &= L^{\pm}\otimes\tau^{-1/2} + 1\otimes L^{\pm}, \\
\Delta(L^3) &= L^3\otimes\tau^{-1/2} + \tau^{1/2}\otimes L^3 \\
&\quad + \lambda\tau^{1/2}(qL^+\otimes L^- + q^{-1}L^-\otimes L^+),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&S(L^{\pm}) = q^{\mp 2}S^{-1}(L^{\pm}) = -L^{\pm}\tau^{1/2}, \\
&S(L^3) = S^{-1}(L^3) = -q^{-2}L^3 + \lambda\lambda_+\tau^{1/2}L^+L^-,
\end{aligned} \tag{24}$$

$$\varepsilon(L^A) = 0, \quad A\in\{+,3,-\}, \tag{25}$$

where $\lambda_+ \equiv q + q^{-1}$. In addition to this, let us notice that $\tau^{1/2}$ is a grouplike generator.

With this Hopf structure at hand we are in a position to write down expressions for q-commutators. Specifically, we get from Eq. (9) for left-commutators

$$\begin{aligned}
[L^{\pm}, V]_q &= (L^{\pm}V - VL^{\pm})\tau^{1/2}, \\
[L^3, V]_q &= L^3V\tau^{1/2} - q^{-2}\tau^{1/2}VL^3 \\
&\quad - \lambda\tau^{1/2}(q^{-1}L^+V\tau^{1/2}L^- + qL^-V\tau^{1/2}L^+) \\
&\quad + \lambda\lambda_+\tau^{1/2}V\tau^{1/2}L^+L^-,
\end{aligned} \tag{26}$$

and likewise for right-commutators,

$$\begin{aligned}
[V, L^{\pm}]_q &= \tau^{1/2}(VL^{\pm} - L^{\pm}V) \\
[V, L^3]_q &= \tau^{1/2}VL^3 - q^{-2}L^3V\tau^{1/2} \\
&\quad - \lambda\tau^{1/2}(q^{-1}L^+V\tau^{1/2}L^- + qL^-V\tau^{1/2}L^+) \\
&\quad + \lambda\lambda_+\tau^{1/2}L^+L^-V\tau^{1/2},
\end{aligned} \tag{27}$$

where V denotes an element living in a representation space of $U_q(su_2)$.

3.3. Matrix representations of $U_q(su_2)$ and commutation relations with tensor operators

Next, we would like to turn our attention to some special representations of the symmetry generators L^+, L^3, and L^-, namely spinor and vector representations [2,22]. The finite dimensional representations of $U_q(su_2)$ are already well known (see for example Refs. [22] and [2]). With our conventions (see also Appendix A) the spinor representations on symmetry generators become

$$\begin{aligned}
&L^A \triangleright \theta^{\alpha} = (\sigma^A)^{\alpha}{}_{\beta}\theta^{\beta}, \quad \tau^{1/2}\triangleright\theta^{\alpha} = (\tau^{1/2})^{\alpha}{}_{\beta}\theta^{\beta}, \\
&\theta_{\alpha}\triangleleft L^A = \theta_{\beta}\,(\sigma^A)^{\beta}{}_{\alpha}, \quad \theta_{\alpha}\triangleleft\tau^{1/2} = \theta_{\beta}\,(\tau^{1/2})^{\beta}{}_{\alpha},
\end{aligned} \tag{28}$$

where we have introduced as some kinds of q-deformed sigma matrices

$$(\sigma^+)^\alpha{}_\beta = -q^{1/2}\lambda_+^{-1/2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad (\sigma^-)^\alpha{}_\beta = q^{-1/2}\lambda_+^{-1/2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$
$$(\sigma^3)^\alpha{}_\beta = \lambda_+^{-1}\begin{pmatrix}-q & 0\\ 0 & q^{-1}\end{pmatrix}, \quad (\tau^{1/2})^\alpha{}_\beta = \begin{pmatrix}q & 0\\ 0 & q^{-1}\end{pmatrix}. \tag{29}$$

Here and in what follows, we shall take the convention that upper and lower matrix indices refer to columns and rows, respectively. That the above matrices indeed give a representation can easily be checked. Towards this end, we have to substitute in (22) the sigma matrices for the algebra generators. Then we can show by usual matrix multiplication that the algebra relations are fulfilled.

The above results enable us to write down commutation relations between symmetry generators and components of a spinor operator. Such a spinor operator with components θ^α, $\alpha = 1, 2$, is completely determined by its transformation properties

$$[L^A, \theta^\alpha]_q = L^A \triangleright \theta^\alpha, \quad [\theta_\alpha, L^A]_q = \theta_\alpha \triangleleft L^A. \tag{30}$$

Inserting the results of (28) and (29) into the relations of (30) and then rearranging, it follows that

$$\begin{aligned}
L^+\theta^1 &= \theta^1 L^+ - q^{1/2}\lambda_+^{-1/2}\theta^2\tau^{-1/2},\\
L^+\theta^2 &= \theta^2 L^+,\\
L^3\theta^1 &= q\theta^1 L^3 - q^{-1/2}\lambda\lambda_+^{-1/2}\theta^2 L^- - q\lambda_+^{-1}\theta^1\tau^{-1/2},\\
L^3\theta^2 &= q^{-1}\theta^2 L^3 + q^{1/2}\lambda\lambda_+^{-1/2}\theta^1 L^+ + q^{-1}\lambda_+^{-1}\theta^2\tau^{-1/2},\\
L^-\theta^1 &= \theta^1 L^-,\\
L^-\theta^2 &= \theta^2 L^- + q^{-1/2}\lambda_+^{-1/2}\theta^1\tau^{-1/2},
\end{aligned} \tag{31}$$

and likewise,

$$\begin{aligned}
\theta_1 L^+ &= L^+\theta_1,\\
\theta_2 L^+ &= L^+\theta_2 - q^{1/2}\lambda_+^{-1/2}\tau^{-1/2}\theta_1,\\
\theta_1 L^3 &= qL^3\theta_1 + q^{-1/2}\lambda\lambda_+^{-1/2}L^+\theta_2 + q\lambda_+^{-1}\tau^{-1/2}\theta_1,\\
\theta_2 L^3 &= q^{-1}L^3\theta_2 - q^{1/2}\lambda\lambda_+^{-1/2}L^-\theta_1 + q^{-1}\lambda_+^{-1}\tau^{-1/2}\theta_2,\\
\theta_1 L^- &= L^-\theta_1 + q^{-1/2}\lambda_+^{-1/2}\tau^{-1/2}\theta_2,\\
\theta_2 L^- &= L^-\theta_2.
\end{aligned} \tag{32}$$

Next, we turn to the vector representations of $U_q(su_2)$. They take the form

$$L^A \triangleright X^B = (\tau^A)^B{}_C X^C, \quad X_B \triangleleft L^A = X_C\,(\tau^A)^C{}_B, \tag{33}$$

with

$$(\tau^+)^B{}_C = \begin{pmatrix}0 & -q & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}, \quad (\tau^-)^B{}_C = \begin{pmatrix}0 & 0 & 0\\ -1 & 0 & 0\\ 0 & q^{-1} & 0\end{pmatrix}, \tag{34}$$

$$(\tau^3)^B{}_C = \begin{pmatrix} q^{-1} & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -q \end{pmatrix}, \quad (\tau^{1/2})^B{}_C = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix},$$

where rows and columns are arranged in the order $+$, 3, and $-$ (from left to right and top to bottom). Their property of being a representation can be verified in very much the same way as was done for the spinor representation. Finally, let us note that the above representation matrices and the q-deformed ε-tensor are linked via

$$(\tau^A)^B{}_C = q^2 \varepsilon^{AB}{}_C, \quad A \in \{+, 3, -\}. \tag{35}$$

With the same reasonings as were already applied to the case of spinor representations we can derive commutation relations between symmetry generators and the components of a vector operator. In complete accordance with the spinor case, their general form is given by

$$[L^A, X^B]_q = L^A \triangleright X^B, \quad [X_B, L^A]_q = X_B \triangleleft L^A, \tag{36}$$

from which we find by specifying q-commutator and vector representation

$$\begin{aligned} L^\pm X^\pm &= X^\pm L^\pm, \\ L^\pm X^\mp &= X^\mp L^\pm \mp X^3 \tau^{-1/2}, \\ L^\pm X^3 &= X^3 L^\pm \mp q^{\pm 1} X^\pm \tau^{-1/2}, \\ L^3 X^\pm &= q^{\mp 2} X^\pm L^3 \pm q^{\mp 1} \lambda X^3 L^\pm \pm q^{\mp 1} X^\pm \tau^{-1/2}, \\ L^3 X^3 &= X^3 L^3 + \lambda (X^- L^+ - X^+ L^-) - \lambda X^3 \tau^{-1/2}, \end{aligned} \tag{37}$$

and

$$\begin{aligned} X_\mp L^\pm &= L^\pm X_\mp, \\ X_\pm L^\pm &= L^\pm X_\pm \mp q^{\pm 1} \tau^{-1/2} X_3, \\ X_3 L^\pm &= L^\pm X_3 \mp \tau^{-1/2} X_\mp, \\ X_\pm L^3 &= q^{\mp 2} L^3 X_\pm \mp \lambda L^\mp X^3 \pm q^{\mp 1} \tau^{-1/2} X_\pm, \\ X_3 L^3 &= L^3 X_3 + \lambda (q^{-1} L^+ X^+ - q L^- X_-) - \lambda \tau^{-1/2} X_3. \end{aligned} \tag{38}$$

In this way, we regain the commutation relations already presented in [26].

3.4. Quantum Lie algebra of $U_q(su_2)$ and its Casimir operator

Recalling that L^A, $A \in \{+, 3, -\}$, is a vector operator, the identities in (36) also apply to the case with X^A being replaced by L^A. In doing this, we are led to the relations for the quantum Lie algebra of $U_q(su_2)$, i.e.

$$[L^A, L^B]_q = q^2 \varepsilon^{AB}{}_C L^C, \tag{39}$$

or more concretely,

$$\begin{aligned} [L^A, L^A]_q &= 0, \qquad A \in \{+, -\}, \\ [L^3, L^3]_q &= -\lambda L^3, \\ [L^\pm, L^3]_q &= \mp q^{\pm 1} L^\pm, \\ [L^\pm, L^\mp]_q &= \mp L^3. \end{aligned} \tag{40}$$

These relations are equivalent to those in (22), as can be seen in a straightforward manner by writing out the q-commutators explicitly.

Instead of the vector operator L^A, one can just as well use a complete antisymmetric tensor operator of second rank given by

$$M^{AB} \equiv \varepsilon^{AB}{}_C L^C \,. \tag{41}$$

Its antisymmetry requires the following identities, which can easily be read off from its very definition, to hold:

$$\begin{aligned} &M^{++} = M^{--} = 0, \quad M^{\pm 3} = -q^{\pm 2} M^{3\pm}, \\ &M^{-+} = -M^{+-}, \quad M^{33} = \lambda M^{+-}. \end{aligned} \tag{42}$$

Consequently, we have three independent components, for which we can choose

$$M^{3\pm} = -q^{\pm 1} L^{\pm}, \quad M^{+-} = -q^{-2} L^3. \tag{43}$$

Last but not least we want to deal with the Casimir operator of our quantum Lie algebra. Let us recall that a Casimir operator C has to be subject to

$$[L^A, C]_q = [C, L^A]_q = 0 \quad \text{for all } A \in \{+, 3, -\}\,. \tag{44}$$

By a direct calculation making use of

$$\begin{aligned} &[L^A, UV]_q = [L^A_{(1)}, U]_q [L^A_{(2)}, V]_q, \\ &[UV, L^A]_q = [U, L^A_{(2)}]_q [V, L^A_{(1)}]_q, \end{aligned} \tag{45}$$

it can be seen that a solution to (44) is given by the expression

$$L^2 \equiv g_{AB} L^A L^B = -q L^+ L^- + L^3 L^3 - q^{-1} L^- L^+, \tag{46}$$

which, in fact, is a q-analog for the Casimir operator of classical angular momentum algebra. Equivalently, the Casimir operator for $U_q(su_2)$ can also be written in terms of the M's. In this manner, it becomes

$$\begin{aligned} M^2 &\equiv g_{AB} g_{CD} M^{AC} M^{BD} \\ &= q^{-1} M^{+3} M^{3-} - M^{+-} M^{+-} + q^{-3} M^{3-} M^{+3}, \end{aligned} \tag{47}$$

which agrees with L^2 up to a constant factor.

Before closing this section, we wish to specify our Casimir operator for the different representations addressed so far. Substituting for the symmetry generators their representations we finally obtain

(a) (operator representation)

$$L^2 = -(X \circ X)(\partial \circ \partial) + q^2 (X \circ \partial)(X \circ \partial) + q^{-2} X \circ \partial, \tag{48}$$

with $U \circ V \equiv g_{AB} U^A V^B$,

(b) (spinor representation)

$$L^2 = q^{-2} \lambda_+^{-2} [[3]]_{q^2} \mathbb{1}_{2\times 2}, \tag{49}$$

(c) (vector representation)

$$L^2 = q^{-2}[[2]]_{q^4}\mathbb{1}_{3\times 3}, \tag{50}$$

where the antisymmetric q-numbers are defined by

$$[[n]]_q \equiv \frac{1-q^{an}}{1-q^a}, \quad n \in \mathbb{N},\ a \in \mathbb{C}. \tag{51}$$

4. Quantum Lie algebra of four-dimensional angular momentum

As the next example we would like to consider the q-deformed algebra of four-dimensional angular momentum. This case can be treated in very much the same way as the three-dimensional one. Thus, we restrict ourselves to stating the results.

4.1. Representation of four-dimensional angular momentum within q-deformed differential calculus

First of all, let us start with the defining representation. Within the differential calculus the generators of q-deformed four-dimensional angular momentum are represented by (see for example Ref. [35])

$$L^{ij} \equiv -q^{-2}\lambda_+\Lambda^{1/2}(P_A)^{ij}{}_{kl}X^k\,\hat{\partial}^l = \lambda_+\Lambda^{1/2}(P_A)^{ij}{}_{kl}\hat{\partial}^k\,X^l, \tag{52}$$

or

$$L^{ij} \equiv -\lambda_+\Lambda^{-1/2}(P_A)^{ij}{}_{kl}X^k\,\partial^l = q^{-2}\lambda_+\Lambda^{-1/2}(P_A)^{ij}{}_{kl}\partial^k\,X^l, \tag{53}$$

where P_A is a q-analog of an antisymmetrizer and Λ denotes a scaling operator subject to

$$\Lambda X^i = q^2X^i\Lambda, \quad \Lambda\partial^i = q^{-2}\partial^i\Lambda. \tag{54}$$

Eq. (53) shows us that the L^{ij} are components of an antisymmetric tensor operator. More specifically, they satisfy

$$\begin{aligned} &L^{ii} = 0, \quad i = 1,\dots,4,\\ &L^{j1} = -qL^{1j}, \quad L^{4j} = -qL^{j4}, \quad j = 2,3,\\ &L^{41} = -L^{14}, \quad L^{32} = -L^{23} + \lambda L^{14}. \end{aligned} \tag{55}$$

Thus, we have in complete analogy to the classical case only six independent generators, for which we can choose the set of L^{ij} with $i < j$. Taking the explicit form of P_A into consideration, we get for them more explicitly

$$\begin{aligned} L^{1i} &= -\Lambda^{-1/2}(q^{-1}X^1\hat{\partial}^i - X^i\hat{\partial}^1), \quad i = 1,2,\\ L^{i4} &= -\Lambda^{-1/2}(q^{-1}X^i\hat{\partial}^4 - X^4\hat{\partial}^i),\\ L^{14} &= 2\lambda_+^{-1}\Lambda^{-1/2}(X^1\hat{\partial}^4 - X^4\hat{\partial}^1)\\ &\quad - \lambda\lambda_+^{-1}\Lambda^{-1/2}(X^2\hat{\partial}^3 + X^3\hat{\partial}^2),\\ L^{23} &= (q^2+q^{-2})\lambda_+^{-1}\Lambda^{-1/2}X^2\hat{\partial}^3\\ &\quad - 2\lambda_+^{-1}\Lambda^{-1/2}X^3\hat{\partial}^2 + \lambda\lambda_+^{-1}\Lambda^{-1/2}(X^1\hat{\partial}^4 - X^4\hat{\partial}^1). \end{aligned} \tag{56}$$

Together with two grouplike braiding operators $K_i, i = 1, 2$, the generators L^{kl}, $k < l$, span the Hopf algebra $U_q(so_4)$. Applying the commutation relations for coordinates, partial derivatives and the two braiding operators (see for example Appendix A and Ref. [1]) leaves us with the nontrivial relations

$$L^{14}L^{1j} - L^{1j}L^{14} = -K_i L^{12}, \quad (i, j) \in \{(2, 2), (1, 3)\}, \tag{57}$$
$$L^{l4}L^{14} - L^{14}L^{l4} = -q^{-2}K_k L^{l4}, \quad (k, l) \in \{(1, 2), (2, 3)\},$$

$$L^{23}L^{12} - L^{12}L^{23} = -qK_2L^{12}, \quad L^{23}L^{13} - L^{13}L^{23} = q^{-1}K_1L^{13}, \tag{58}$$
$$L^{24}L^{23} - L^{23}L^{24} = q^{-3}K_1L^{24}, \quad L^{34}L^{23} - L^{23}L^{34} = -q^{-1}K_2L^{34},$$

$$K_1L^{13} = q^{-2}L^{13}K_1, \quad K_2L^{12} = q^{-2}L^{12}K_2, \tag{59}$$
$$K_1L^{24} = q^2L^{24}K_1, \quad K_2L^{34} = q^2L^{34}K_2.$$

4.2. Hopf structure of $U_q(so_4)$ and corresponding q-commutators

As in the three-dimensional case we need the Hopf structure to proceed further. For the independent generators this reads as (see also Ref. [35])

$$\Delta(L^{1j}) = L^{1j} \otimes K_i + 1 \otimes L^{12}, \quad (i, j) \in \{(2, 2), (1, 3)\}, \tag{60}$$
$$\Delta(L^{l4}) = L^{l4} \otimes K_k + 1 \otimes L^{l4}, \quad (k, l) \in \{(1, 2), (2, 3)\},$$
$$\begin{aligned}\Delta(L^{14}) = {} & q\lambda_+^{-1}L^{14} \otimes K_1 + q^{-1}\lambda_+^{-1}L^{14} \otimes K_2\\ & + q\lambda_+^{-1}K_1^{-1} \otimes L^{14} + q^{-1}\lambda_+^{-1}K_2^{-1} \otimes L^{14}\\ & - \lambda_+^{-1}L^{23} \otimes K_1 + \lambda_+^{-1}L^{23} \otimes K_2\\ & - \lambda_+^{-1}K_1^{-1} \otimes L^{23} + \lambda_+^{-1}K_2^{-1} \otimes L^{23}\\ & - q\lambda\lambda_+^{-1}L^{24}K_1^{-1} \otimes L^{13} - q\lambda\lambda_+^{-1}K_1^{-1}L^{13} \otimes L^{24}\\ & - q\lambda\lambda_+^{-1}L^{34}K_2^{-1} \otimes L^{12} - q\lambda\lambda_+^{-1}K_2^{-1}L^{12} \otimes L^{34},\end{aligned}$$
$$\begin{aligned}\Delta(L^{23}) = {} & -\lambda_+^{-1}K_1^{-1} \otimes L^{14} + \lambda_+^{-1}K_2^{-1} \otimes L^{14}\\ & - \lambda_+^{-1}L^{14} \otimes K_1 + \lambda_+^{-1}L^{14} \otimes K_2\\ & + q^{-1}\lambda_+^{-1}L^{23} \otimes K_1 + q\lambda_+^{-1}L^{23} \otimes K_2\\ & + q^{-1}\lambda_+^{-1}K_1^{-1} \otimes L^{23} + q\lambda_+^{-1}K_2^{-1} \otimes L^{23}\\ & + \lambda\lambda_+^{-1}L^{24}K_1^{-1} \otimes L^{13} + \lambda\lambda_+^{-1}K_1^{-1}L^{13} \otimes L^{24}\\ & - q^2\lambda\lambda_+^{-1}L^{34}K_2^{-1} \otimes L^{12} - q^2\lambda\lambda_+^{-1}K_2^{-1}L^{12} \otimes L^{34},\end{aligned}$$

$$S(L^{1j}) = q^{-2}S^{-1}(L^{1j}) = -L^{1j}K_i^{-1}, \quad (i, j) \in \{(2, 2), (1, 3)\}, \tag{61}$$
$$S(L^{l4}) = q^2S^{-1}(L^{l4}) = -L^{l4}K_k^{-1}, \quad (k, l) \in \{(1, 2), (2, 3)\},$$
$$S(L^{14}) = S^{-1}(L^{14}) = L^{14} - q^{-1}\lambda^{-1}((K_1 + K_2) - (K_1^{-1} + K_2^{-1})),$$
$$S(L^{23}) = S^{-1}(L^{23}) = L^{23} + q^{-1}\lambda^{-1}(q^{-1}(K_1 - K_1^{-1}) - q(K_2 - K_2^{-1})),$$
$$\varepsilon(L^{mn}) = 0. \tag{62}$$

Again, this Hopf structure enables us to introduce q-commutators in complete analogy to the three-dimensional case. Explicitly writing these out we obtain for the q-commutator with an element living in a representation space of $U_q(so_4)$

$$[L^{1j}, V]_q = (L^{1j}V - VL^{1j})K_i^{-1}, \quad (i, j) \in \{(2, 2), (1, 3)\}, \tag{63}$$

$$
\begin{aligned}
[L^{l4}, V]_q &= (L^{l4}V - VL^{l4})K_k^{-1}, \quad (k,l) \in \{(1,2),(2,3)\},\\
[L^{14}, V]_q \;&=\; -q^{-1}\lambda^{-1}(K_1^{-1}VK_1 + K_2^{-1}VK_2)\\
&\quad + q^{-1}\lambda^{-1}(K_1^{-1}VK_1^{-1} + K_2^{-1}VK_2^{-1})\\
&\quad - \lambda_+^{-1}(K_2^{-1}VL^{23} + L^{23}VK_2^{-1} - K_1^{-1}VL^{23} + L^{23}VK_1^{-1})\\
&\quad + q^{-1}\lambda_+^{-1}(K_2^{-1}VL^{14} + L^{14}VK_2^{-1})\\
&\quad + q\lambda_+^{-1}(K_1^{-1}VL^{14} + L^{14}VK_1^{-1})\\
&\quad + q\lambda\lambda_+^{-1}(K_1^{-1}L^{13}VL^{24}K_1^{-1} + K_2^{-1}L^{12}VL^{34}K_2^{-1})\\
&\quad + q\lambda\lambda_+^{-1}(L^{24}K_1^{-1}VL^{13}K_1^{-1} + L^{34}K_2^{-1}VL^{12}K_2^{-1}),\\
[L^{23}, V]_q \;&=\; q\lambda_+^{-1}(K_2^{-1}VL^{23} + L^{23}VK_2^{-1})\\
&\quad + q^{-1}\lambda_+^{-1}(K_1^{-1}VL^{23} + L^{23}VK_1^{-1})\\
&\quad + \lambda_+^{-1}(L^{14}VK_2^{-1} + K_2^{-1}VL^{14} - L^{14}VK_1^{-1} - K_1^{-1}VL^{14})\\
&\quad + \lambda^{-1}(K_2^{-1}VK_2^{-1} - K_2^{-1}VK_2)\\
&\quad + q^{-2}\lambda^{-1}(K_1^{-1}VK_1 - K_1^{-1}VK_1^{-1})\\
&\quad + q^2\lambda\lambda_+^{-1}(K_2^{-1}L^{12}VL^{34}K_2^{-1} + L^{34}K_2^{-1}VL^{12}K_2^{-1})\\
&\quad - \lambda\lambda_+^{-1}(K_1^{-1}L^{13}VL^{24}K_1^{-1} - L^{24}K_1^{-1}VL^{13}K_1^{-1}),
\end{aligned}
$$

and likewise for their right versions,

$$
\begin{aligned}
[V, L^{1j}]_q &= K_i^{-1}(VL^{1j} - L^{1j}V), \quad (i,j) \in \{(2,2),(1,3)\}, \qquad (64)\\
[V, L^{l4}]_q &= K_k^{-1}(VL^{l4} - L^{l4}V), \quad (k,l) \in \{(1,2),(2,3)\},\\
[V, L^{14}]_q \;&=\; q\lambda_+^{-1}(L^{14}VK_1^{-1} + K_1^{-1}VL^{14})\\
&\quad + q^{-1}\lambda_+^{-1}(L^{14}VK_2^{-1} + K_2^{-1}VL^{14})\\
&\quad + \lambda_+^{-1}(L^{23}VK_2^{-1} + K_2^{-1}VL^{23} - L^{23}VK_1^{-1} - K_1^{-1}VL^{23})\\
&\quad + q^{-1}\lambda^{-1}(K_1^{-1}VK_1^{-1} - K_1VK_1^{-1})\\
&\quad + q^{-1}\lambda^{-1}(K_2^{-1}VK_2^{-1} - K_2VK_2^{-1})\\
&\quad + q\lambda\lambda_+^{-1}(K_1^{-1}L^{13}VL^{24}K_1^{-1} + K_1^{-1}L^{24}VK_1^{-1}L^{13})\\
&\quad + q\lambda\lambda_+^{-1}(K_2^{-1}L^{34}VK_2^{-1}L^{12} + K_2^{-1}L^{12}VL^{34}K_2^{-1}),\\
[V, L^{23}]_q \;&=\; q\lambda_+^{-1}(L^{23}VK_2^{-1} + K_2^{-1}VL^{23})\\
&\quad + q^{-1}\lambda_+^{-1}(L^{23}VK_1^{-1} + K_1^{-1}VL^{23})\\
&\quad + \lambda_+^{-1}(L^{14}VK_2^{-1} + K_2^{-1}VL^{14} - L^{14}VK_1^{-1} - K_1^{-1}VL^{14})\\
&\quad + \lambda^{-1}(K_2^{-1}VK_2^{-1} - K_2VK_2^{-1})\\
&\quad + q^{-2}\lambda^{-1}(K_1VK_1^{-1} - K_1^{-1}VK_1^{-1})\\
&\quad + \lambda\lambda_+^{-1}(q^2K_2^{-1}L^{34}VK_2^{-1}L^{12} + q^2K_2^{-1}L^{12}VL^{34}K_2^{-1})\\
&\quad - \lambda\lambda_+^{-1}(K_1^{-1}L^{13}VL^{24}K_1^{-1} - K_1^{-1}L^{24}VK_1^{-1}L^{13}).
\end{aligned}
$$

4.3. Matrix representations of $U_q(so_4)$ and commutation relations with tensor operators

Let us now go on to the spinor and vector representations of the L^{ij}. As in the classical case, we can distinguish two types of spinor representations, i.e. (1/2, 0) and (0, 1/2) (see for example Ref. [35]). Explicitly, we have

$$
L^{ij} \triangleright \theta^\alpha = (\sigma^{ij})^\alpha{}_\beta \theta^\beta, \quad L^{ij} \triangleright \tilde{\theta}^\alpha = (\tilde{\sigma}^{ij})^\alpha{}_\beta\, \tilde{\theta}^\beta \qquad (65)
$$

$$\theta_\alpha \triangleleft L^{ij} = \theta_\beta (\sigma^{ij})^\beta{}_\alpha, \quad \tilde{\theta}_\alpha \triangleleft L^{ij} = \tilde{\theta}_\beta (\tilde{\sigma}^{ij})^\beta{}_\alpha,$$

with

$$(\sigma^{13})^\alpha{}_\beta = \begin{pmatrix} 0 & -q^{-2} \\ 0 & 0 \end{pmatrix}, \qquad (\sigma^{24})^\alpha{}_\beta = \begin{pmatrix} 0 & 0 \\ -q^{-1} & 0 \end{pmatrix}, \tag{66}$$
$$(\sigma^{14})^\alpha{}_\beta = -q(\sigma^{23})^\alpha{}_\beta = q^{-1}\lambda_+^{-1} \begin{pmatrix} -q & 0 \\ 0 & q^{-1} \end{pmatrix},$$
$$(\sigma^{34})^\alpha{}_\beta = (\sigma^{12})^\alpha{}_\beta = 0.$$

The matrices with tildes one gets most easily from the identities

$$\begin{aligned} (\tilde{\sigma}^{13})^\alpha{}_\beta &= (\sigma^{12})^\alpha{}_\beta, \quad (\tilde{\sigma}^{12})^\alpha{}_\beta = (\sigma^{13})^\alpha{}_\beta, \\ (\tilde{\sigma}^{14})^\alpha{}_\beta &= (\sigma^{14})^\alpha{}_\beta, \quad (\tilde{\sigma}^{23})^\alpha{}_\beta = -q^{-2}(\sigma^{23})^\alpha{}_\beta, \\ (\tilde{\sigma}^{34})^\alpha{}_\beta &= (\sigma^{24})^\alpha{}_\beta, \quad (\tilde{\sigma}^{24})^\alpha{}_\beta = (\sigma^{34})^\alpha{}_\beta. \end{aligned} \tag{67}$$

Notice that the tilde on the spinor components will remind us of the fact that they transform differently. That these matrices determine a representation of $U_q(so_4)$ can again be proven by inserting them together with

$$(K_1)^\alpha{}_\beta = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad (\tilde{K}_2)^\alpha{}_\beta = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \tag{68}$$
$$(\tilde{K}_1)^\alpha{}_\beta = (K_2)^\alpha{}_\beta = \mathbb{1},$$

into relations (57)–(59).

From the spinor representation of $U_q(so_4)$ we can – as usual – compute commutation relations between the L^{ij} and the components of a spinor operator. To achieve this we apply the identities

$$\left[L^{ij}, \theta^\alpha\right]_q = (\sigma^{ij})^\alpha{}_\beta \theta^\beta, \quad \left[\theta^\alpha, L^{ij}\right]_q = \theta_\beta\, (\sigma^{ij})^\beta{}_\alpha\,. \tag{69}$$

If we write out the q-commutators and substitute the representation matrices without tildes for the L^{ij} we get the commutation relations

$$\begin{aligned} L^{13}\theta^2 &= \theta^2 L^{13} - q^{-2}\theta^1 K_1, \\ L^{24}\theta^1 &= \theta^1 L^{24} - q^{-1}\theta^2 K_1, \\ L^{14}\theta^1 &= (q^2 + q^{-1})\lambda_+^{-1}\theta^1 L^{14} - (q-1)\lambda_+^{-1}\theta^1 L^{23} \\ &\quad + q\lambda\lambda_+^{-1}\theta^2 L^{13} - \lambda_+^{-1}\theta^1 K_1, \\ L^{14}\theta^2 &= (q^{-1} + 1)\lambda_+^{-1}\theta^2 L^{14} + (q^{-1} - 1)\lambda_+^{-1}\theta^2 L^{23} \\ &\quad + \lambda\lambda_+^{-1}\theta^1 L^{24} + q^{-2}\lambda_+^{-1}\theta^2 K_1, \\ L^{23}\theta^1 &= (q+1)\lambda_+^{-1}\theta^1 L^{23} - (q-1)\lambda_+^{-1}\theta^1 L^{14} \\ &\quad - \lambda\lambda_+^{-1}\theta^2 L^{13} + q^{-1}\lambda_+^{-1}\theta^1 K_1, \\ L^{23}\theta^2 &= (q^{-2} + q)\lambda_+^{-1}\theta^2 L^{23} + (1 - q^{-1})\lambda_+^{-1}\theta^2 L^{14} \\ &\quad - q^{-1}\lambda\lambda_+^{-1}\theta^1 L^{24} - q^{-3}\lambda_+^{-1}\theta^2 K_1, \end{aligned} \tag{70}$$

and

$$
\begin{aligned}
\theta_1 L^{13} &= L^{13}\theta_1 - q^{-2}K_1\theta_2, \\
\theta_2 L^{24} &= L^{24}\theta_2 - q^{-1}K_1\theta_1, \\
\theta_1 L^{14} &= (q^2+q^{-1})\lambda_+^{-1}L^{14}\theta_1 - (q-1)\lambda_+^{-1}L^{23}\theta_1 \\
&\quad + \lambda\lambda_+^{-1}L^{24}\theta_2 - \lambda_+^{-1}K_1\theta_1, \\
\theta_2 L^{14} &= (q^{-1}+1)\lambda_+^{-1}L^{14}\theta_2 + (1-q^{-1})\lambda_+^{-1}L^{23}\theta_2 \\
&\quad + q\lambda\lambda_+^{-1}L^{13}\theta_1 + q^{-2}\lambda_+^{-1}K_1\theta_2, \\
\theta_1 L^{23} &= (q+1)\lambda_+^{-1}L^{23}\theta_1 - (q-1)\lambda_+^{-1}L^{14}\theta_1 \\
&\quad - q^{-1}\lambda\lambda_+^{-1}L^{24}\theta_2 + q^{-1}\lambda_+^{-1}K_1\theta_1, \\
\theta_2 L^{23} &= (q+q^{-2})\lambda_+^{-1}L^{23}\theta_2 + (1-q^{-1})\lambda_+^{-1}L^{14}\theta_2 \\
&\quad - \lambda\lambda_+^{-1}L^{13}\theta_1 - q^{-3}\lambda_+^{-1}K_1\theta_2.
\end{aligned}
\tag{71}
$$

The remaining relations are trivial, i.e. these relations take the form $L^{ij}\theta^k = \theta^k L^{ij}$. Repeating the same steps as before for spinors (and the representation matrices) with tildes we arrive at the following nontrivial relations:

$$
\begin{aligned}
L^{12}\tilde{\theta}^2 &= \tilde{\theta}^2 L^{12} - q^{-2}\tilde{\theta}^1 K_2, \\
L^{34}\tilde{\theta}^1 &= \tilde{\theta}^1 L^{34} - q^{-1}\tilde{\theta}^2 K_2, \\
L^{14}\tilde{\theta}^1 &= (q+1)\lambda_+^{-1}\tilde{\theta}^1 L^{14} + (q-1)\lambda_+^{-1}\tilde{\theta}^1 L^{23} \\
&\quad + q\lambda\lambda_+^{-1}\tilde{\theta}^2 L^{12} - \lambda_+^{-1}\tilde{\theta}^1 K_2, \\
L^{14}\tilde{\theta}^2 &= (q^{-2}+q)\lambda_+^{-1}\tilde{\theta}^2 L^{12} + (q^{-1}-1)\lambda_+^{-1}\tilde{\theta}^2 L^{23} \\
&\quad + \lambda\lambda_+^{-1}\tilde{\theta}^1 L^{34} + q^{-2}\lambda_+^{-1}\tilde{\theta}^2 K_2, \\
L^{23}\tilde{\theta}^1 &= (q^{-1}+q^2)\lambda_+^{-1}\tilde{\theta}^1 L^{23} + (q-1)\lambda_+^{-1}\tilde{\theta}^1 L^{14} \\
&\quad + q^2\lambda\lambda_+^{-1}\tilde{\theta}^2 L^{12} - q\lambda_+^{-1}\tilde{\theta}^1 K_2, \\
L^{23}\tilde{\theta}^2 &= (q^{-1}+1)\lambda_+^{-1}\tilde{\theta}^2 L^{23} + (q^{-1}-1)\lambda_+^{-1}\tilde{\theta}^2 L^{14} \\
&\quad + q\lambda\lambda_+^{-1}\tilde{\theta}^1 L^{34} + q^{-1}\lambda_+^{-1}\tilde{\theta}^2 K_2,
\end{aligned}
\tag{72}
$$

and

$$
\begin{aligned}
\tilde{\theta}_1 L^{12} &= L^{12}\tilde{\theta}_1 - q^{-2}K_2\tilde{\theta}_2, \\
\tilde{\theta}_2 L^{34} &= L^{34}\tilde{\theta}_2 - q^{-1}K_2\tilde{\theta}_1, \\
\tilde{\theta}_1 L^{14} &= (q+1)\lambda_+^{-1}L^{14}\tilde{\theta}_1 + (q-1)\lambda_+^{-1}L^{23}\tilde{\theta}_1 \\
&\quad + \lambda\lambda_+^{-1}L^{34}\tilde{\theta}_2 - \lambda_+^{-1}K_2\tilde{\theta}_1, \\
\tilde{\theta}_2 L^{14} &= (q+q^{-2})\lambda_+^{-1}L^{14}\tilde{\theta}_2 + (q^{-1}-1)\lambda_+^{-1}L^{23}\tilde{\theta}_2 \\
&\quad + q\lambda\lambda_+^{-1}L^{12}\tilde{\theta}_1 + q^{-2}\lambda_+^{-1}K_2\tilde{\theta}_2, \\
\tilde{\theta}_1 L^{23} &= (q^2+q^{-1})\lambda_+^{-1}L^{23}\tilde{\theta}_1 + (q-1)\lambda_+^{-1}L^{14}\tilde{\theta}_1 \\
&\quad + q\lambda\lambda_+^{-1}L^{34}\tilde{\theta}_2 - q\lambda_+^{-1}K_2\tilde{\theta}_1, \\
\tilde{\theta}_2 L^{23} &= (1+q^{-1})\lambda_+^{-1}L^{23}\tilde{\theta}_2 + (q^{-1}-1)\lambda_+^{-1}L^{14}\tilde{\theta}_2 \\
&\quad + q^2\lambda\lambda_+^{-1}L^{12}\tilde{\theta}_1 + q^{-1}\lambda_+^{-1}K_2\tilde{\theta}_2.
\end{aligned}
\tag{73}
$$

These considerations carry over to the vector representations of the independent generators of $U_q(so_4)$. Its right and left versions are given by

$$L^{ij} \triangleright X^k = (\tau^{ij})^k{}_m X^m, \quad X_k \triangleleft L^{ij} = X_m (\tau^{ij})^m{}_k, \tag{74}$$

with

$$(\tau^{14})^k{}_m = q^{-1}\lambda_+^{-1}\begin{pmatrix} -2q & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 2q^{-1} \end{pmatrix}, \tag{75}$$

$$(\tau^{12})^k{}_m = q^{-2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\tau^{13})^k{}_m = q^{-2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(\tau^{24})^k{}_m = q^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (\tau^{34})^k{}_m = q^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$(\tau^{23})^k{}_m = q^{-1}\lambda_+^{-1}\begin{pmatrix} -q\lambda & 0 & 0 & 0 \\ 0 & -(q^2+q^{-2}) & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & q^{-1}\lambda \end{pmatrix}.$$

If we want to check that the above matrices give a representation of $U_q(so_4)$ we additionally need the matrices corresponding to the braiding operators, i.e.

$$(K_1)^k{}_m = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad (K_2)^k{}_m = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{76}$$

We are again in a position to write down commutation relations between the independent L^{ij} and the components of a vector operator. Starting from

$$[L^{ij}, X^k]_q = (\tau^{ij})^k{}_l X^l, \tag{77}$$

and proceeding in very much the same way as in the spinor case we obtain the relations

$$\begin{aligned} L^{1j}X^1 &= X^1 L^{1j}, \\ L^{1j}X^j &= X^j L^{1j}, \\ L^{1k}X^l &= X^l L^{1k} - q^{-2}X^1 K_i, \\ L^{1m}X^4 &= X^4 L^{1m} + q^{-2}X^m K_n, \end{aligned} \tag{78}$$

$$\begin{aligned} L^{m'4}X^1 &= X^1 L^{m'4} - q^{-1}X^{m'} K_n, \\ L^{j4}X^j &= X^j L^{j4}, \\ L^{l4}X^k &= X^k L^{l4} + q^{-1}X^4 K_i, \\ L^{j4}X^4 &= X^4 L^{j4}, \end{aligned} \tag{79}$$

and

$$\begin{aligned} L^{14}X^1 &= qX^1L^{14} - \lambda_+^{-1}X^1(K_1+K_2) \\ &\quad + q\lambda\lambda_+^{-1}(X^2L^{13} + X^3L^{12}), \\ L^{14}X^2 &= 2\lambda_+^{-1}X^2L^{14} + \lambda_+^{-1}X^2(q^2K_1 - K_2) + \lambda\lambda_+^{-1}X^2L^{23} \\ &\quad + \lambda\lambda_+^{-1}(X^1L^{24} - qX^4L^{12}), \\ L^{14}X^3 &= q^{-2}(q^4+1)\lambda_+^{-1}X^3L^{14} + \lambda_+^{-1}X^3(q^{-2}K_2 - K_1) \\ &\quad - \lambda\lambda_+^{-1}X^3L^{23} + \lambda\lambda_+^{-1}(X^1L^{34} - qX^4L^{13}), \\ L^{14}X^4 &= q^{-1}X^4L^{14} + q^{-2}\lambda_+^{-1}X^4(K_1+K_2) \\ &\quad - \lambda\lambda_+^{-1}(X^2L^{34} + X^3L^{24}), \end{aligned} \tag{80}$$

$$\begin{aligned} L^{23}X^1 &= qX^1L^{23} + \lambda_+^{-1}X^1(q^{-1}K_1 - qK_2) \\ &\quad - \lambda\lambda_+^{-1}(X^2L^{13} - q^2X^3L^{12}), \\ L^{23}X^2 &= q^{-2}(q^4+1)X^2L^{23} - \lambda_+^{-1}X^2(q^{-3}K_1 + qK_2) \\ &\quad - q^2\lambda_+^{-1}X^4L^{12} - \lambda\lambda_+^{-1}(q^{-1}X^1L^{24} + X^2L^{14}), \\ L^{23}X^3 &= 2\lambda_+^{-1}X^3L^{23} + q^{-1}\lambda_+^{-1}X^3(K_1+K_2) \\ &\quad - \lambda\lambda_+^{-1}X^3L^{14} + \lambda\lambda_+^{-1}(qX^1L^{34} + X^4L^{13}), \\ L^{23}X^4 &= q^{-1}X^4L^{23} - q^{-1}\lambda_+^{-1}X^4(q^{-2}K_1 - K_2) \\ &\quad - \lambda\lambda_+^{-1}(qX^2L^{34} - q^{-1}X^3L^{24}), \end{aligned} \tag{81}$$

where

$$\begin{aligned} &j = 2,3, \quad (k,l,i) \in \{(2,3,2),(3,2,1)\}, \\ &(m,n) \in \{(2,2),(3,1)\}, \quad m' \equiv 5 - m'. \end{aligned} \tag{82}$$

The right version of (77), i.e.

$$[X_k, L^{ij}]_q = X_l\,(\tau^{ij})^l{}_k, \tag{83}$$

gives us likewise

$$\begin{aligned} X_1L^{1m} &= L^{1m}X_1 - q^{-2}K_nX_3, \\ X_{l'}L^{1k} &= L^{1k}X_{l'} + q^{-2}K_iX_4, \\ X_{j'}L^{1j} &= L^{1j}X_{j'}, \\ X_4L^{1j} &= L^{1j}X_4, \end{aligned} \tag{84}$$

$$\begin{aligned} X_1L^{j4} &= L^{j4}X_1, \\ X_{k'}L^{l4} &= L^{l4}X_2 - q^{-1}K_iX_1, \\ X_{j'}L^{j4} &= L^{j4}X_{j'}, \\ X_4L^{m'4} &= L^{m'4}X_4 + q^{-1}K_nX_m, \end{aligned} \tag{85}$$

and

$$\begin{aligned} X_1L^{14} &= qL^{14}X_1 - \lambda_+^{-1}(K_1+K_2)X_1 \\ &\quad + \lambda\lambda_+^{-1}(L^{24}X_2 + L^{34}X_3), \end{aligned} \tag{86}$$

$$
\begin{aligned}
X_2 L^{14} &= 2\lambda_+^{-1} L^{14} X_2 + \lambda_+^{-1}(q^{-2}K_1 - K_2)X_2 \\
&\quad + \lambda\lambda_+^{-1} L^{23} X_2 - \lambda\lambda_+^{-1}(L^{34}X_4 + qL^{13}X_1), \\
X_3 L^{14} &= q^{-2}(q^4+1)\lambda_+^{-1} L^{14} X_3 + \lambda_+^{-1}(q^{-2}K_2 - K_1)X_3 \\
&\quad - \lambda\lambda_+^{-1} L^{23} X_3 + \lambda\lambda_+^{-1}(qL^{12}X_1 - L^{24}X_4), \\
X_4 L^{14} &= q^{-1} L^{14} X_4 + q^{-2}\lambda_+^{-1}(K_1 + K_2)X_4 \\
&\quad - q\lambda\lambda_+^{-1}(L^{12}X_2 + L^{13}X_3), \\
X_1 L^{23} &= qL^{23}X_1 + q^{-1}\lambda_+^{-1}(K_1 - q^2K_2)X_1 \\
&\quad + \lambda\lambda_+^{-1}(qL^{34}X_3 - q^{-1}L^{24}X_2), \qquad (87) \\
X_2 L^{23} &= q^{-2}(q^4+1)L^{23}X_2 - \lambda_+^{-1}(q^{-3}K_1 + qK_2)X_2 \\
&\quad + \lambda\lambda_+^{-1} L^{14} X_2 - \lambda\lambda_+^{-1}(L^{13}X_1 + qL^{34}X_4), \\
X_3 L^{23} &= 2\lambda_+^{-1} L^{23} X_3 + q^{-1}\lambda_+^{-1}(K_1 + K_2)X_3 \\
&\quad - \lambda\lambda_+^{-1} L^{14} X_3 + \lambda\lambda_+^{-1}(q^2L^{12}X_1 + q^{-1}L^{24}X_4), \\
X_4 L^{23} &= q^{-1} L^{23} X_4 + q^{-1}\lambda_+^{-1}(K_2 - q^{-2}K_1)X_4 \\
&\quad + \lambda\lambda_+^{-1}(L^{13}X_3 - q^2L^{12}X_2).
\end{aligned}
$$

4.4. Quantum Lie algebra of $U_q(so_4)$ and its Casimir operators

The L^{ij} can act on themselves through the adjoint representation, which can be obtained via the following simple reasoning. We calculate the action of the L^{ij} on a product of antisymmetrized vector coordinates (for their definition see Ref. [34]), i.e.

$$L^{ij} \triangleright \xi^k \xi^l = \left((L^{ij})_{(1)} \triangleright \xi^k\right)\left((L^{ij})_{(2)} \triangleright \xi^l\right) = \sum_{mn} (a^{ij,kl})_{mn}\, \xi^m \xi^n. \qquad (88)$$

As already mentioned, the L^{ij} are components of an antisymmetric tensor operator. Hence, they have to act on themselves in the same way as they act on a product of antisymmetrized coordinates. In this sense we read off from (88) for their quantum Lie algebra

$$[L^{ij}, L^{kl}]_q = L^{ij} \triangleright L^{kl} = \sum_{mn} (a^{ij,kl})_{mn}\, L^{mn}. \qquad (89)$$

Explicitly, the non-vanishing q-commutators for the independent L^{ij} become

$$
\begin{aligned}
&[L^{12}, L^{23}]_q = -q^{-2}[L^{23}, L^{12}]_q = q^{-1}L^{12}, \qquad (90)\\
&[L^{12}, L^{34}]_q = -[L^{34}, L^{12}]_q = -q^{-2}L^{23} - q^{-3}L^{14}, \\
&[L^{13}, L^{23}]_q = -q^{-2}[L^{23}, L^{13}]_q = -q^{-3}L^{13}, \\
&[L^{13}, L^{24}]_q = -[L^{24}, L^{13}]_q = q^{-2}L^{23} - q^{-1}L^{14}, \\
&[L^{14}, L^{1i}]_q = -q^2[L^{1i}, L^{14}]_q = -L^{1i}, \quad i = 2, 3, \qquad (91)\\
&[L^{14}, L^{14}]_q = -q^{-1}\lambda L^{14}, \\
&[L^{14}, L^{23}]_q = [L^{23}, L^{14}]_q = -q^{-1}\lambda L^{23}, \\
&[L^{14}, L^{i4}]_q = -q^{-2}[L^{i4}, L^{14}]_q = q^{-2}L^{i4}, \quad i = 2, 3, \\
&[L^{23}, L^{23}]_q = -q^{-1}\lambda L^{14} - q^{-1}\lambda^2 L^{23}, \qquad (92)
\end{aligned}
$$

$$[L^{23}, L^{24}]_q = -q^{-2}[L^{24}, L^{23}]_q = -q^{-3}L^{24},$$
$$[L^{23}, L^{34}]_q = -q^{-2}[L^{34}, L^{23}]_q = q^{-1}L^{34}.$$

Let us remark that these identities can be checked by means of the spinor and vector representations of the L^{ij}. Towards this end, one has to write out the q-commutators using (63) or (64). Then we replace the generators by their representation matrices and apply usual matrix multiplication. Proceeding in this manner will again show us the validity of the above identities.

Finally, let us consider the Casimir operators, which can be introduced as in the three-dimensional case. The first one is given by the expression

$$\begin{aligned} C_1 &= g_{ik}g_{jm}L^{ij}L^{km} = 2L^{23}L^{23} + \lambda_+(L^{12}L^{34} + L^{13}L^{24}) \\ &= +q^2\lambda_+(L^{24}L^{13} + L^{34}L^{12}) + (q^2 + q^{-2})L^{14}L^{14} \\ &\quad -\lambda(L^{14}L^{23} + L^{23}L^{14}), \end{aligned} \tag{93}$$

and the second one by

$$\begin{aligned} C_2 = \varepsilon_{ijkl}L^{ij}L^{kl} &= q^2\lambda_+^2(L^{14}L^{23} + L^{23}L^{14}) + q^2\lambda_+^2(L^{12}L^{34} - L^{13}L^{24}) \\ &\quad + q^4\lambda_+^2(L^{34}L^{12} - L^{24}L^{13}) - q^2\lambda\lambda_+^2 L^{14}L^{14}, \end{aligned} \tag{94}$$

where g_{ik} and ε_{ijkl} denote respectively the quantum metric and the q-deformed epsilon tensor (for its definition see Ref. [17]) corresponding to $U_q(so_4)$. Again, we would like to specify the Casimir operators for the different representations. In doing so, we get the following:

(a) (operator representation)

$$\begin{aligned} C_1 &= 2q^{-2}(X \circ X)(\partial \circ \partial) + 2q^2(X \circ \partial)(X \circ \partial) \\ &\quad + 2q^{-1}\lambda_+ X \circ \partial, \end{aligned} \tag{95}$$
$$C_2 = 0,$$

with $U \circ V \equiv g_{AB}U^A V^B$,

(b) (spinor representation)

$$C_1 = [[3]]_{q^{-4}}\mathbb{1}_{2\times 2}, \tag{96}$$
$$C_2 = q[[3]]_{q^4}\lambda_+\mathbb{1}_{2\times 2},$$

(c) (vector representation)

$$C_1 = 2[[3]]_{q^{-4}}\mathbb{1}_{3\times 3}, \tag{97}$$
$$C_2 = 0.$$

5. Quantum Lie algebra of Lorentz transformations

In this section we deal with the quantum Lie algebra of Lorentz transformations. Everything so far applies to this case, which from a physical point of view is the most interesting one we consider in this article.

5.1. Representation of Lorentz generators within q-deformed differential calculus

In complete analogy to the previous section we start with a realization of the $V^{\mu\nu}$ given by [26]

$$V^{\mu\nu} \equiv \Lambda^{1/2}(P_A)^{\mu\nu}{}_{\rho\sigma}X^\rho\,\hat{\partial}^\sigma = -q^{-2}\Lambda^{1/2}(P_A)^{\mu\nu}{}_{\rho\sigma}\hat{\partial}^\rho X^\sigma, \tag{98}$$

or equivalently

$$V^{\mu\nu} \equiv q^{-2}\Lambda^{-1/2}(P_A)^{\mu\nu}{}_{\rho\sigma}X^{\rho}\,\partial^{\sigma} = -\Lambda^{-1/2}(P_A)^{\mu\nu}{}_{\rho\sigma}\partial^{\rho}\,X^{\sigma}, \tag{99}$$

where Λ is a scaling operator subject to

$$\Lambda X^{\mu} = q^{-2}X^{\mu}\Lambda, \quad \Lambda\partial^{\mu} = q^{2}\partial^{\mu}\Lambda, \tag{100}$$

and P_A denotes the antisymmetrizer for q-deformed Lorentz symmetry. More explicitly, we have for example

$$\begin{aligned}
V^{+3} &= 2q^2\lambda_+^{-2}X^+\hat{\partial}^3 - 2\lambda_+^{-2}X^3\hat{\partial}^+ \\
&\quad - q\lambda\lambda_+^{-2}(X^+\hat{\partial}^0 - X^0\hat{\partial}^+), \\
V^{+0} &= 2\lambda_+^{-2}X^+\hat{\partial}^0 - q^{-2}(q^4+1)\lambda_+^{-2}X^0\hat{\partial}^+ \\
&\quad + \lambda\lambda_+^{-2}(qX^+\hat{\partial}^3 - q^{-1}X^3\hat{\partial}^+), \\
V^{+-} &= 2\lambda_+^{-2}(\lambda^2+1)X^+\hat{\partial}^- - 2\lambda_+^{-2}X^-\hat{\partial}^+ \\
&\quad - \lambda\lambda_+^{-2}(X^3\hat{\partial}^0 + X^0\hat{\partial}^3), \\
V^{30} &= 2\lambda_+^{-2}X^3\hat{\partial}^0 - q^{-2}(q^4+1)\lambda_+^{-2}X^0\hat{\partial}^3 \\
&\quad + \lambda\lambda_+^{-2}(\lambda^2+1)(X^+\hat{\partial}^- - X^-\hat{\partial}^+), \\
V^{3-} &= 2q^2\lambda_+^{-2}X^3\hat{\partial}^- - 2\lambda_+^{-2}X^-\hat{\partial}^3 \\
&\quad - q\lambda\lambda_+^{-2}(X^0\hat{\partial}^- + X^-\hat{\partial}^0), \\
V^{0-} &= 2\lambda_+^{-2}X^0\hat{\partial}^- - q^{-2}(q^4+1)\lambda_+^{-2}X^-\hat{\partial}^0 \\
&\quad - \lambda\lambda_+^{-2}(qX^3\hat{\partial}^- - q^{-1}X^-\hat{\partial}^3).
\end{aligned} \tag{101}$$

The antisymmetry of the $V^{\mu\nu}$ implies that we have six independent components for which we can choose

$$V^{+3}, V^{+0}, V^{+-}, V^{30}, V^{3-}, V^{0-}, \tag{102}$$

since the remaining ones are related to them by the relations

$$\begin{aligned}
&V^{++} = V^{00} = V^{--} = 0, \\
&V^{33} = \lambda V^{+-}, \quad V^{-+} = -V^{+-}, \\
&V^{\pm 3} = -q^{\pm 2}V^{3\pm}, \\
&V^{\pm 0} = \mp q^{\pm 1}\lambda V^{3\pm} - V^{0\pm}, \\
&V^{03} = \lambda V^{+-} - V^{30}.
\end{aligned} \tag{103}$$

In Ref. [26] it was shown that the $V^{\mu\nu}$ together with two additional generators, denoted in the following by U^1, U^2, span q-deformed Lorentz algebra. However, in what follows it is convenient to introduce another set of generators which are related to the $V^{\mu\nu}$ by

$$\begin{aligned}
R^+ &= q^{-1}\lambda_+^{-1}(V^{+0} - V^{+3}), \\
R^3 &= q^{-1}\lambda_+^{-1}(V^{30} - qV^{+-}), \\
R^- &= q^{-1}\lambda_+^{-1}(q^{-2}V^{3-} + V^{0-}),
\end{aligned} \tag{104}$$

$$S^+ = q^{-1}\lambda_+^{-1}(V^{+0} + q^{-2}V^{+3}), \tag{105}$$

$$S^3 = q^{-1}\lambda_+^{-1}(V^{30} + q^{-1}V^{+-}),$$
$$S^- = q^{-1}\lambda_+^{-1}(V^{3-} - V^{0-}).$$

In terms of the generators R^A and S^A the relations of q-deformed Lorentz algebra take the rather compact form

$$\begin{aligned} &\epsilon_{CB}{}^{A} R^B R^C = q^{-4}\lambda_+^{-1} U^1 R^A, \\ &\epsilon_{CB}{}^{A} S^B S^C = -q^{-4}\lambda_+^{-1} U^2 S^A, \\ &R^A S^B = q^2 (\hat{R}_{SO_q(3)})^{AB}_{CD} S^C R^D, \end{aligned} \tag{106}$$

while U^1 and U^2 are both central in the algebra. $\epsilon_{CB}{}^{A}$ in (106) denotes the epsilon tensor from Section 3 and $\hat{R}_{SO_q(3)}$ stands for the vector representation of the universal R-matrix of $SO_q(3)$. For more details we refer the reader to [26].

5.2. Hopf structure of q-deformed Lorentz algebra and corresponding q-commutators

To write down q-commutators with Lorentz generators $V^{\mu\nu}$ we need to know their Hopf structure. From Ref. [40], however, we know the coproduct for the generators R^A and S^A, $A \in \{+, 3, -\}$. By solving (104) and (105) for the independent generators given in (102) and making use of the algebra homomorphism property of the coproduct we are able to find expressions for the coproducts of the $V^{\mu\nu}$.

For compactness, we introduce

$$\rho \equiv q^2\lambda\lambda_+ R^3 + U^1, \quad \sigma \equiv q^2\lambda\lambda_+ S^3 - U^2, \tag{107}$$

and the generators of an $U_q(su_2)$-subalgebra, given by [40]

$$\begin{aligned} &L^A \equiv -q^2\lambda_+(U^1 S^A - U^2 R^A + q^2\lambda\lambda_+ \epsilon_{CB}{}^{A} R^B S^C), \\ &\tau^{-1/2} \equiv U^1 U^2 - q^4\lambda\lambda_+ g_{AB} R^A S^B + \lambda L^3, \end{aligned} \tag{108}$$

where g_{AB} is the quantum metric of Section 3. In terms of the new quantities the coproducts wanted read as

$$\begin{aligned} \Delta(V^{+3}) = &-q^2\sigma \otimes R^+ + q^2 R^+ \otimes \rho \\ &- q^2\tau^{1/2} R^+ \otimes \tau^{1/2}\rho + q^2\tau^{1/2}\sigma \otimes \tau^{1/2} R^+ \\ &- q^3\lambda_+^{-1}\tau^{1/2}\sigma L^+ \otimes \sigma - q^3\lambda_+^{-1}\tau^{1/2}\sigma \otimes \tau^{1/2}\sigma L^+ \\ &+ q^7\lambda^2\lambda_+^2\tau^{1/2} R^+ \otimes \tau^{1/2} S^- L^+ + q^9\lambda^2\lambda_+^2\tau^{1/2} R^+ L^+ \otimes S^-, \\ \Delta(V^{+0}) = &\ \sigma \otimes R^+ - R^+ \otimes \rho \\ &- q^2\tau^{1/2} R^+ \otimes \tau^{1/2}\rho + q^2\tau^{1/2}\sigma \otimes \tau^{1/2} R^+ \\ &- q^3\lambda_+^{-1}\tau^{1/2}\sigma L^+ \otimes \sigma - q^3\lambda_+^{-1}\tau^{1/2}\sigma \otimes \tau^{1/2}\sigma L^+ \\ &+ q^7\lambda^2\lambda_+^2\tau^{1/2} R^+ \otimes \tau^{1/2} S^- L^+ + q^9\lambda^2\lambda_+^2\tau^{1/2} R^+ L^+ \otimes S^-, \\ \Delta(V^{+-}) = &\ q R^3 \otimes \rho - q\tau^{1/2}\sigma \otimes R^3 \\ &+ q S^3 \otimes \sigma - q\tau^{1/2}\rho \otimes S^3 \\ &- q^4\lambda\lambda_+ S^- \otimes R^+ - q^2\lambda\lambda_+ R^+ \otimes S^- \\ &+ q^4\lambda\lambda_+\tau^{1/2} R^+ \otimes R^- + q^3\lambda\tau^{1/2}\sigma L^- \otimes R^+ \\ &+ q^2\lambda\lambda_+\tau^{1/2} S^- \otimes S^+ + q^5\lambda\tau^{1/2}\rho L^+ \otimes S^-, \end{aligned} \tag{109}$$

$$\begin{aligned}\Delta(V^{30}) \;=\; & -R^3 \otimes \rho + \tau^{1/2}\sigma \otimes R^3 \\ & + q^2 S^3 \otimes \sigma - q^2 \tau^{1/2} \rho \otimes S^3 \\ & + q^3 \lambda\lambda_+ S^- \otimes R^+ - q^3 \lambda\lambda_+ R^+ \otimes S^- \\ & - q^3 \lambda\lambda_+ \tau^{1/2} R^+ \otimes R^- - q^2 \lambda \tau^{1/2} \sigma L^- \otimes R^+ \\ & + q^3 \lambda\lambda_+ \tau^{1/2} S^- \otimes S^+ + q^6 \lambda \tau^{1/2} \rho L^+ \otimes S^-, \\ \Delta(V^{3-}) \;=\; & q^2 S^- \otimes \sigma - q^2 \rho \otimes S^- \\ & - q^2 \tau^{1/2} S^- \otimes \tau^{1/2}\sigma + q^2 \tau^{1/2} \rho \otimes \tau^{1/2} S^- \\ & - q^3 \lambda_+^{-1} \tau^{1/2} \rho L^- \otimes \rho - q^3 \lambda_+^{-1} \tau^{1/2} \rho \otimes \tau^{1/2} \rho L^- \\ & + q^5 \lambda^2 \lambda_+^2 \tau^{1/2} S^- L^- \otimes R^+ + q^7 \lambda^2 \lambda_+^2 \tau^{1/2} S^- \otimes \tau^{1/2} R^+ L^-, \\ \Delta(V^{0-}) \;=\; & -S^- \otimes \sigma + \rho \otimes S^- \\ & - q^2 \tau^{1/2} S^- \otimes \tau^{1/2}\sigma + q^2 \tau^{1/2} \rho \otimes \tau^{1/2} S^- \\ & - q^3 \lambda_+^{-1} \tau^{1/2} \rho L^- \otimes \rho - q^3 \lambda_+^{-1} \tau^{1/2} \rho \otimes \tau^{1/2} \rho L^- \\ & + q^5 \lambda^2 \lambda_+^2 \tau^{1/2} S^- L^- \otimes R^+ + q^7 \lambda^2 \lambda_+^2 \tau^{1/2} S^- \otimes \tau^{1/2} R^+ L^-. \end{aligned}$$

For writing down q-commutators we also need the following antipodes and their inverses:

$$S(R^+) = q^2 S^{-1}(R^+) = -q^2 \tau^{1/2} R^+, \tag{110}$$
$$S(R^3) = S^{-1}(R^3) = -q^{-2} \lambda^{-1} \lambda_+^{-1} (U^1 + \tau^{1/2})\sigma,$$
$$S(R^-) = q^{-2} S^{-1}(R^-) = -S^- - q^{-1} \lambda_+^{-1} \tau^{1/2} L^- \sigma,$$
$$S(S^+) = q^2 S^{-1}(S^+) = -R^+ - q^3 \lambda_+^{-1} \tau^{1/2} L^+ \rho, \tag{111}$$
$$S(S^3) = S^{-1}(S^3) = q^{-2} \lambda^{-1} \lambda_+^{-1} (U^2 - \tau^{1/2}\rho),$$
$$S(S^-) = q^{-2} S^{-1}(S^-) = -q^{-2} \tau^{1/2} S^-,$$
$$S(U^i) = S^{-1}(U^i) = U^i, \quad i = 1, 2, \tag{112}$$
$$S(\rho) = S^{-1}(\rho) = -\tau^{1/2}\sigma, \tag{113}$$
$$S(\sigma) = S^{-1}(\sigma) = -\tau^{1/2}\rho.$$

After these preparations we are ready to write down q-commutators. For their left versions we have found

$$\begin{aligned}[V^{+3}, V]_q \;=\; & q^3 \lambda_+^{-1} \tau^{1/2} \sigma (L^+ V - V L^+) \tau^{1/2} \rho \\ & + q^7 \lambda^2 \lambda_+^2 \tau^{1/2} R^+ (V L^+ - L^+ V) \tau^{1/2} S^- \\ & + q^2 (q^2 \sigma V \tau^{1/2} R^+ - \tau^{1/2} \sigma V R^+) \\ & + q^2 (\tau^{1/2} R^+ V \sigma - R^+ V \tau^{1/2} \sigma), \end{aligned} \tag{114}$$

$$\begin{aligned}[V^{+0}, V]_q \;=\; & q^3 \lambda_+^{-1} \tau^{1/2} \sigma (L^+ V - V L^+) \tau^{1/2} \rho \\ & + q^7 \lambda^2 \lambda_+^2 \tau^{1/2} R^+ (V L^+ - L^+ V) \tau^{1/2} S^- \\ & - q^2 (\sigma V \tau^{1/2} R^+ + \tau^{1/2} \sigma V R^+) \\ & + q^2 \tau^{1/2} R^+ V \sigma + R^+ V \tau^{1/2} \sigma, \end{aligned} \tag{115}$$

$$\begin{aligned}[V^{+-},V]_q = {} & q(R^3V\tau^{1/2}\sigma + S^3V\tau^{1/2}\rho) \\ & +q^{-1}\lambda^{-1}\lambda_+^{-1}\tau^{1/2}(\sigma V(U^1+\tau^{1/2}\sigma) - \rho V(U^2-\tau^{1/2}\rho)) \\ & -q^3\lambda\tau^{1/2}(\rho L^+V\tau^{1/2}S^- + q^2S^-V\tau^{1/2}L^+\rho) \\ & -q^3\lambda\tau^{1/2}(q^2\sigma L^-V\tau^{1/2}R^+ + R^+V\tau^{1/2}L^-\sigma) \\ & +q^4\lambda\lambda_+(q^2S^-V\tau^{1/2}R^+ - \tau^{1/2}R^+VS^-) \\ & +\lambda\lambda_+(R^+V\tau^{1/2}S^- - q^2\tau^{1/2}S^-VR^+), \end{aligned} \tag{116}$$

$$\begin{aligned}[V^{30},V]_q = {} & R^3V\tau^{1/2}\sigma - q^2S^3V\tau^{1/2}\rho \\ & -q^{-2}\lambda^{-1}\lambda_+^{-1}\tau^{1/2}(\sigma V(U^1+\tau^{1/2}\sigma) + q^2\rho V(U^2-\tau^{1/2}\rho)) \\ & -q^4\lambda\tau^{1/2}(\rho L^+V\tau^{1/2}S^- + q^2S^-V\tau^{1/2}L^+\rho) \\ & +q^2\lambda\tau^{1/2}(q^2\sigma L^-V\tau^{1/2}R^+ + R^+V\tau^{1/2}L^-\sigma) \\ & +q^3\lambda\lambda_+(q^2S^-V\tau^{1/2}R^+ - \tau^{1/2}R^+VS^-) \\ & +q\lambda\lambda_+(R^+V\tau^{1/2}S^- - q^2\tau^{1/2}S^-VR^+), \end{aligned} \tag{117}$$

$$\begin{aligned}[V^{3-},V]_q = {} & \rho V\tau^{1/2}S^- - q^2\tau^{1/2}\rho VS^- \\ & +q^2(\tau^{1/2}S^-V\rho - S^-V\tau^{1/2}\rho) \\ & -q^3\,\lambda_+^{-1}\tau^{1/2}\rho(L^-V - VL^-)\tau^{1/2}\sigma \\ & -q^7\lambda^2\lambda_+^2\tau^{1/2}S^-(VL^- - L^-V)\tau^{1/2}R^+, \end{aligned} \tag{118}$$

$$\begin{aligned}[V^{0-},V]_q = {} & -q^{-2}\rho V\tau^{1/2}S^- - q^2\tau^{1/2}\rho VS^- \\ & +q^2\tau^{1/2}S^-V\rho + S^-V\tau^{1/2}\rho \\ & -q^3\lambda_+^{-1}\tau^{1/2}\rho(L^-V - VL^-)\tau^{1/2}\sigma \\ & -q^7\lambda^2\lambda_+^2\tau^{1/2}S^-(VL^- - L^-V)\tau^{1/2}R^+, \end{aligned} \tag{119}$$

and likewise for the corresponding right versions,

$$\begin{aligned}[V,V^{+3}]_q = {} & q^2\sigma V\tau^{1/2}R^+ - q^2\sigma\tau^{1/2}VR^+ \\ & -R^+V\tau^{1/2}\sigma + R^+\tau^{1/2}V\sigma \\ & -q^3\,\lambda_+^{-1}(\rho V\tau^{1/2}\sigma L^+ - L^+\rho V\tau^{1/2}\sigma) \\ & +q^9\lambda^2\lambda_+^2\tau^{1/2}(L^+S^-V\tau^{1/2}R^+ - q^2S^-V\tau^{1/2}R^+L^+), \end{aligned} \tag{120}$$

$$\begin{aligned}[V,V^{+0}]_q = {} & q^2\sigma V\tau^{1/2}R^+ + \sigma\tau^{1/2}VR^+ \\ & -R^+V\tau^{1/2}\sigma - q^{-2}R^+\tau^{1/2}V\sigma \\ & -q^3\,\lambda_+^{-1}(\rho V\tau^{1/2}\sigma L^+ - L^+\rho V\tau^{1/2}\sigma) \\ & +q^9\lambda^2\lambda_+^2\tau^{1/2}(L^+S^-V\tau^{1/2}R^+ - q^2S^-V\tau^{1/2}R^+L^+), \end{aligned} \tag{121}$$

$$\begin{aligned}[V,V^{+-}]_q = {} & -q(\rho\tau^{1/2}VS^3 + \sigma\tau^{1/2}VR^3) \\ & +q^{-1}\lambda^{-1}\lambda_+^{-1}((-U^2+\rho\tau^{1/2})V\tau^{1/2}\rho + (U^1+\sigma\tau^{1/2})V\tau^{1/2}\sigma) \\ & -q\lambda(L^+\rho\tau^{1/2}V\tau^{1/2}S^- + q^6S^-\tau^{1/2}V\tau^{1/2}\rho L^+) \\ & -q\lambda(q^4\tau^{1/2}L^-\sigma V\tau^{1/2}R^+ + R^+\tau^{1/2}V\tau^{1/2}\sigma L^-) \\ & +\lambda\lambda_+(q^4S^-\tau^{1/2}VR^+ - R^+V\tau^{1/2}S^-) \\ & +q^2\lambda\lambda_+(R^+\tau^{1/2}VS^- - q^4S^-V\tau^{1/2}R^+), \end{aligned} \tag{122}$$

$$\begin{aligned}[V,V^{30}]_q = {} & \sigma\tau^{1/2}VR^3 - q^2\rho\tau^{1/2}VS^3 \\ & - q^{-2}\lambda^{-1}\lambda_+^{-1}((U^1+\sigma\tau^{1/2})V\tau^{1/2}\sigma + q^2(U^2-\rho\tau^{1/2})V\tau^{1/2}\rho) \\ & + \lambda(R^+\tau^{1/2}V\tau^{1/2}\sigma L^- + q^4\tau^{1/2}L^-\sigma V\tau^{1/2}R^+) \\ & - q^2\lambda(L^+\rho\tau^{1/2}V\tau^{1/2}S^- - q^6S^-\tau^{1/2}V\tau^{1/2}\rho L^+) \\ & - q\lambda\lambda_+(R^+V\tau^{1/2}S^- + R^+\tau^{1/2}VS^-) \\ & + q^5\lambda\lambda_+(S^-V\tau^{1/2}R^+ + S^-\tau^{1/2}VR^+), \end{aligned} \tag{123}$$

$$\begin{aligned}[V,V^{3-}]_q = {} & -q^2\rho\tau^{1/2}VS^- + q^2\rho V\tau^{1/2}S^- \\ & - q^4S^-V\tau^{1/2}\rho + q^4S^-\tau^{1/2}V\rho \\ & + q^3\lambda_+^{-1}(\sigma\tau^{1/2}V\tau^{1/2}\rho L^- - \tau^{1/2}L^-\sigma V\tau^{1/2}\rho) \\ & + q^3\lambda^2\lambda_+^2(q^2\tau^{1/2}L^-R^+V\tau^{1/2}S^- - R^+\tau^{1/2}V\tau^{1/2}S^-L^-), \end{aligned} \tag{124}$$

$$\begin{aligned}[V,V^{0-}]_q = {} & \rho\tau^{1/2}VS^- + q^2\rho V\tau^{1/2}S^- \\ & - q^4S^-V\tau^{1/2}\rho - q^2S^-\tau^{1/2}V\rho \\ & + q^3\lambda_+^{-1}(\sigma\tau^{1/2}V\tau^{1/2}\rho L^- - \tau^{1/2}L^-\sigma V\tau^{1/2}\rho) \\ & - q^3\lambda^2\lambda_+^2(q^2\tau^{1/2}L^-R^+V\tau^{1/2}S^- - R^+\tau^{1/2}V\tau^{1/2}S^-L^-). \end{aligned} \tag{125}$$

5.3. Matrix representations of q-deformed Lorentz algebra and commutation relations with tensor operators

Let us now consider spin representations of q-deformed Lorentz algebra. From a physical point of view q-deformed analogs of the two spinor representations $(1/2, 0)$ and $(0, 1/2)$ as well as the vector representation $(1/2, 1/2)$ are the most interesting cases [6,25,37,42]. For a systematic treatment of finite dimensional representations of q-Lorentz algebra we refer the reader to for example [4]. As in the previous sections, we begin with spinor representations, for which we have

$$\begin{aligned} & V^{\mu\nu} \triangleright \theta^\alpha = \left(\sigma^{\mu\nu}\right)^\alpha{}_\beta\,\theta^\beta, \quad V^{\mu\nu} \triangleright \bar\theta^\alpha = \left(\bar\sigma^{\mu\nu}\right)^\alpha{}_\beta\,\bar\theta^\beta, \\ & \theta_\alpha \triangleleft V^{\mu\nu} = \theta_\beta\left(\sigma^{\mu\nu}\right)^\beta{}_\alpha, \quad \bar\theta_\alpha \triangleleft V^{\mu\nu} = \bar\theta_\beta\left(\bar\sigma^{\mu\nu}\right)^\beta{}_\alpha, \end{aligned} \tag{126}$$

and the spin matrices are given by

$$\begin{aligned} & (\sigma^{+3})^\alpha{}_\beta = q^2(\sigma^{+0})^\alpha{}_\beta = -q^{1/2}\lambda_+^{-3/2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \\ & (\sigma^{+-})^\alpha{}_\beta = -q(\sigma^{30})^\alpha{}_\beta = q^{-1}\lambda_+^{-2}\begin{pmatrix}-q & 0\\ 0 & q^{-1}\end{pmatrix}, \\ & (\sigma^{3-})^\alpha{}_\beta = (\sigma^{0-})^\alpha{}_\beta = q^{-1/2}\lambda_+^{-3/2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \end{aligned} \tag{127}$$

in connection with

$$\begin{aligned} & (\bar\sigma^{+3})^\alpha{}_\beta = (\sigma^{+3})^\alpha{}_\beta, \quad (\bar\sigma^{+0})^\alpha{}_\beta = -q^2(\sigma^{+0})^\alpha{}_\beta, \\ & (\bar\sigma^{+-})^\alpha{}_\beta = (\sigma^{+-})^\alpha{}_\beta, \quad (\bar\sigma^{30})^\alpha{}_\beta = -q^2(\sigma^{30})^\alpha{}_\beta, \\ & (\bar\sigma^{3-})^\alpha{}_\beta = (\sigma^{3-})^\alpha{}_\beta, \quad (\bar\sigma^{0-})^\alpha{}_\beta = -q^2(\sigma^{0-})^\alpha{}_\beta. \end{aligned} \tag{128}$$

The spinor representations of the generators U^1, U^2, and σ are given by diagonal matrices which fulfill the identities

$$-q^2(q^2+q^{-2})^{-1}\lambda_+(U^1)^\alpha{}_\beta = -(U^2)^\alpha{}_\beta = (\sigma)^\alpha{}_\beta = \delta^\alpha{}_\beta\,. \tag{129}$$

Furthermore, for ρ and the generators of the $U_q(su_2)$-subalgebra we have

$$(L^+)^\alpha{}_\beta = -q^{1/2}\lambda_+^{-1/2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad (L^-)^\alpha{}_\beta = -q^{-1/2}\lambda_+^{-1/2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \tag{130}$$
$$(\tau^{-1/2})^\alpha{}_\beta = -(\rho)^\alpha{}_\beta = \begin{pmatrix}q^{-1} & 0\\ 0 & q\end{pmatrix}.$$

What we have done so far enables us to write out the commutation relations between Lorentz generators and components of a spinor operator. In general, they are equivalent to

$$[V^{\mu\nu},\theta^\alpha]_q = (\sigma^{\mu\nu})^\alpha{}_\beta\,\theta^\beta, \quad [V^{\mu\nu},\bar\theta^\alpha]_q = (\bar\sigma^{\mu\nu})^\alpha{}_\beta\bar\theta^\beta, \tag{131}$$
$$[\theta_\alpha, V^{\mu\nu}]_q = \theta_\beta\,(\sigma^{\mu\nu})^\beta{}_\alpha, \quad [\bar\theta_\alpha, V^{\mu\nu}]_q = \bar\theta_\beta(\bar\sigma^{\mu\nu})^\beta{}_\alpha\,.$$

Unfortunately, it is rather difficult to derive from (131) the explicit form of the commutation relations between Lorentz generators and spinors. Thus, we proceed differently and apply the identities

$$V^{\mu\nu}a = ((V^{\mu\nu})_{(1)}\triangleright a)(V^{\mu\nu})_{(2)}, \tag{132}$$
$$aV^{\mu\nu} = (V^{\mu\nu})_{(2)}(a\triangleleft (V^{\mu\nu})_{(1)}),$$

which result from Eqs. (7) and (8). In this way the first relation in (132) implies

$$\begin{aligned} V^{+3}\theta^1 \;=\;& (q+1)\lambda_+^{-1}\theta^1V^{+3} + (q^2-q)\lambda_+^{-1}\theta^1V^{+0}\\ &- q^{1/2}\lambda\lambda_+^{-1/2}\theta^2(V^{+-}+qV^{30}) + q^{1/2}\lambda_+^{-3/2}\theta^2\rho,\\ V^{+3}\theta^2 =\;& (q+q^{-2})\lambda_+^{-1}\theta^2V^{+3} + (1-q)\lambda_+^{-1}\theta^2V^{+0}, \end{aligned} \tag{133}$$

$$\begin{aligned} V^{+0}\theta^1 \;=\;& (q^2+q^{-1})\lambda_+^{-1}\theta^1V^{+0} + (1-q^{-1})\lambda_+^{-1}\theta^1V^{+3}\\ &- q^{1/2}\lambda\lambda_+^{-1/2}\theta^2(V^{+-}+qV^{30}) - q^{-3/2}\lambda_+^{-3/2}\theta^2\rho,\\ V^{+0}\theta^2 =\;& (1+q^{-1})\lambda_+^{-1}\theta^2V^{+0} + (q^{-2}-q^{-1})\lambda_+^{-1}\theta^2V^{+3}, \end{aligned} \tag{134}$$

$$\begin{aligned} V^{+-}\theta^1 \;=\;& q^{-1}(2q^2+\lambda_+)\lambda_+^{-2}\theta^1V^{+-} + q^{-1}(q-1)^2\lambda_+^{-2}\theta^1V^{30}\\ &+ q^{1/2}(\lambda_+-1)\lambda\lambda_+^{-3/2}\theta^2V^{0-}\\ &- q^{-3/2}(q^2\lambda_++1)\lambda\lambda_+^{-3/2}\theta^2V^{3-} + \lambda_+^{-2}\theta^1U^1,\\ V^{+-}\theta^2 \;=\;& q^{-1}(2q^2+\lambda_+)\lambda_+^{-2}\theta^2V^{+-} + q^{-1}(q-1)^2\lambda_+^{-2}\theta^2V^{30}\\ &+ q^{-1/2}\lambda\lambda_+^{-3/2}\theta^1(V^{+0}-V^{+3}) - q^{-2}\lambda_+^{-2}\theta^2U^1, \end{aligned} \tag{135}$$

$$\begin{aligned} V^{30}\theta^1 \;=\;& q^{-1}(q^2\lambda_++2)\lambda_+^{-2}\theta^1V^{30} + q^{-1}(q-1)^2\lambda_+^{-2}\theta^1V^{+-}\\ &+ q^{-1/2}(1+q^2\lambda_+)\lambda\lambda_+^{-3/2}\theta^2V^{0-}\\ &+ q^{-5/2}(1-q^4\lambda_+)\lambda\lambda_+^{-3/2}\theta^2V^{3-} - q^{-1}\lambda_+^{-2}\theta^1U^1,\\ V^{30}\theta^2 \;=\;& q^{-1}(q^2\lambda_++2)\lambda_+^{-2}\theta^2V^{30} + q^{-1}(q-1)^2\lambda_+^{-2}\theta^2V^{+-}\\ &+ q^{-3/2}\lambda\lambda_+^{-3/2}\theta^1(V^{+0}-V^{+3}) + q^{-3}\lambda_+^{-2}\theta^2U^1, \end{aligned} \tag{136}$$

$$V^{3-}\theta^1 = (q^{-1}+1)\lambda_+^{-1}\theta^1 V^{3-} + (q-1)\lambda_+^{-1}\theta^2 V^{0-}, \tag{137}$$

$$V^{3-}\theta^2 = (q^2+q^{-1})\lambda_+^{-1}\theta^2 V^{3-} - (q^2-q)\lambda_+^{-1}\theta^1 V^{0-} - q^{-1/2}\lambda_+^{-3/2}\theta^1\rho,$$

$$V^{0-}\theta^1 = (q+q^{-2})\lambda_+^{-1}\theta^1 V^{0-} + (q^{-1}-q^{-2})\lambda_+^{-1}\theta^1 V^{3-}, \tag{138}$$

$$V^{0-}\theta^2 = (q+1)\lambda_+^{-1}\theta^2 V^{0-} + (q^{-1}-1)\lambda_+^{-1}\theta^2 V^{3-} - q^{-1/2}\lambda_+^{-3/2}\theta^1\rho.$$

For the second set of spinors we get from (132)

$$V^{+3}\bar\theta^1 = (q+q^{-1})\lambda_+^{-1}\bar\theta^1 V^{+3} + (q-1)\lambda_+^{-1}\bar\theta^1 V^{+0} - q^{1/2}\lambda_+^{-3/2}\bar\theta^2\sigma, \tag{139}$$

$$V^{+3}\bar\theta^2 = (q^2+q^{-1})\lambda_+^{-1}\bar\theta^2 V^{+3} - (q^2-q)\lambda_+^{-1}\bar\theta^2 V^{+0},$$

$$V^{+0}\bar\theta^1 = (q+q^{-2})\lambda_+^{-1}\bar\theta^1 V^{+0} + (q^{-1}-q^{-2})\lambda_+^{-1}\bar\theta^1 V^{+3} - q^{1/2}\lambda_+^{-3/2}\bar\theta^2\sigma, \tag{140}$$

$$V^{+0}\bar\theta^2 = (q+1)\lambda_+^{-1}\bar\theta^2 V^{+0} - (1-q^{-1})\lambda_+^{-1}\bar\theta^2 V^{+3},$$

$$V^{+-}\bar\theta^1 = q^{-1}(2+q^2\lambda_+)\lambda_+^{-2}\bar\theta^1 V^{+-} - q^{-1}(q-1)^2\lambda_+^{-2}\bar\theta^1 V^{30} + q^{1/2}\lambda\lambda_+^{-3/2}\bar\theta^2(V^{0-}-V^{3-}) + \lambda_+^{-2}\bar\theta^1 U^2, \tag{141}$$

$$V^{+-}\bar\theta^2 = q^{-1}(2+q^2\lambda_+)\lambda_+^{-2}\bar\theta^2 V^{+-} + q^{-1}(q-1)^2\lambda_+^{-2}\bar\theta^2 V^{30} - q^{-1/2}\lambda(\lambda_++1)\lambda_+^{-3/2}\bar\theta^1 V^{+0} + q^{-5/2}\lambda(q^2\lambda_++1)\lambda_+^{-3/2}\bar\theta^1 V^{+3} - q^{-2}\lambda_+^{-2}\bar\theta^2 U^1,$$

$$V^{30}\bar\theta^1 = q^{-1}(\lambda_++2q^2)\lambda_+^{-2}\bar\theta^1 V^{30} - q^{-1}(q-1)^2\lambda_+^{-2}\bar\theta^1 V^{+-} + q^{3/2}\lambda\lambda_+^{-3/2}\bar\theta^2(V^{0-}-V^{3-}) + q\lambda_+^{-2}\bar\theta^1 U^2, \tag{142}$$

$$V^{30}\bar\theta^2 = q^{-1}(\lambda_++2q^2)\lambda_+^{-2}\bar\theta^2 V^{30} - q^{-1}(q-1)^2\lambda_+^{-2}\bar\theta^2 V^{+-} - q^{-3/2}\lambda(\lambda_++1)\lambda_+^{-3/2}\bar\theta^1 V^{+3} + q^{-3/2}\lambda(\lambda_++q^2)\lambda_+^{-3/2}\bar\theta^1 V^{+0} - q^{-1}\lambda_+^{-2}\bar\theta^2 U^2, \tag{143}$$

$$V^{3-}\bar\theta^1 = (q+1)\lambda_+^{-1}\bar\theta^1 V^{3-} + (q^2-q)\lambda_+^{-1}\bar\theta^1 V^{0-}, \tag{144}$$

$$V^{3-}\bar\theta^2 = (q+q^{-2})\lambda_+^{-1}\bar\theta^2 V^{3-} - (q-1)\lambda_+^{-1}\bar\theta^2 V^{0-} + q^{1/2}\lambda\lambda_+^{-1/2}\bar\theta^1(qV^{+-}-V^{30}) + q^{-1/2}\lambda_+^{-3/2}\bar\theta^1\sigma,$$

$$V^{0-}\bar\theta^1 = (q^2+q^{-1})\lambda_+^{-1}\bar\theta^1 V^{0-} + (1-q^{-1})\lambda_+^{-1}\bar\theta^1 V^{3-}, \tag{145}$$

$$V^{0-}\bar\theta^2 = (1+q^{-1})\lambda_+^{-1}\bar\theta^2 V^{0-} - (q^{-1}-q^{-2})\lambda_+^{-1}\bar\theta^2 V^{3-} + q^{1/2}\lambda\lambda_+^{-1/2}\bar\theta^1(qV^{+-}-V^{30}) - q^{-5/2}\lambda_+^{-3/2}\bar\theta^1\sigma.$$

The right versions of the above relations read

$$\theta_1 V^{+3} = (q+1)\lambda_+^{-1}V^{+3}\theta_1 + (q^2-q)\lambda_+^{-1}V^{+0}\theta_1, \tag{146}$$

$$\theta_2 V^{+3} = (q+q^{-2})\lambda_+^{-1}V^{+3}\theta_2 - (q-1)\lambda_+^{-1}V^{+0}\theta_2 - q^{1/2}\lambda\lambda_+^{-1/2}(V^{+-}+qV^{30})\theta_1 + q^{1/2}\lambda_+^{-3/2}\rho\,\theta_1,$$

$$\theta_1 V^{+0} = (q^2+q)\lambda_+^{-1} V^{+0}\theta_1 + (1-q^{-1})\lambda_+^{-1} V^{+3}\theta_1, \tag{147}$$
$$\begin{aligned}\theta_2 V^{+0} \;=\; & (1+q^{-1})\lambda_+^{-1} V^{+0}\theta_2 - (q^{-1}-q^{-2})\lambda_+^{-1} V^{+3}\theta_2 \\ & - q^{1/2}\lambda\lambda_+^{-1/2}(V^{+-}+qV^{30})\theta_1 - q^{-3/2}\lambda_+^{-3/2}\rho\,\theta_1,\end{aligned}$$

$$\begin{aligned}\theta_1 V^{+-} \;=\; & q^{-1}(q^4+1+\lambda_+)\lambda_+^{-2} V^{+-}\theta_1 \\ & - q^{-2}(q^4+1-q^2\lambda_+)\lambda_+^{-2} V^{30}\theta_1 \\ & + q^{-1/2}\lambda\lambda_+^{-3/2}(V^{+3}-V^{+0})\theta_2 \\ & + q^{-1}\lambda\lambda_+^{-2}\rho\,\theta_1 + q^{-2}\lambda_+^{-2}U^1\theta_1,\end{aligned} \tag{148}$$
$$\begin{aligned}\theta_2 V^{+-} \;=\; & q^{-1}(q^4+1+\lambda_+)\lambda_+^{-2} V^{+-}\theta_2 \\ & - q^{-2}(q^4+1-q^2\lambda_+)\lambda_+^{-2} V^{30}\theta_2 \\ & - q^{-3/2}(q^2\lambda_+ + 1)\lambda\lambda_+^{-3/2} V^{3-}\theta_1 \\ & - q^{1/2}(\lambda_+ - 1)\lambda\lambda_+^{-3/2} V^{0-}\theta_1 \\ & + q^{-1}\lambda\lambda_+^{-2}\rho\,\theta_2 - \lambda_+^{-2}U^1\theta_2,\end{aligned}$$

$$\begin{aligned}\theta_1 V^{30} \;=\; & q^{-3}(q^4+1+q^4\lambda_+)\lambda_+^{-2} V^{30}\theta_1 \\ & - q^{-2}(q^4+1-q^2\lambda_+)\lambda_+^{-2} V^{+-}\theta_1 \\ & + q^{-3/2}\lambda\lambda_+^{-3/2}(V^{+0}-V^{+3})\theta_2 \\ & - q^{-2}\lambda\lambda_+^{-2}\rho\,\theta_1 - q^{-3}\lambda_+^{-2}U^1\theta_1,\end{aligned} \tag{149}$$
$$\begin{aligned}\theta_2 V^{30} \;=\; & q^{-3}(q^4+1+q^4\lambda_+)\lambda_+^{-2} V^{30}\theta_2 \\ & - q^{-2}(q^4+1-q^2\lambda_+)\lambda_+^{-2} V^{+-}\theta_2 \\ & + q^{-1/2}(q^2\lambda_+ + 1)\lambda\lambda_+^{-3/2} V^{0-}\theta_1 \\ & + q^{-5/2}(1-q^4\lambda_+)\lambda\lambda_+^{-3/2} V^{3-}\theta_1 \\ & - q^{-2}\lambda\lambda_+^{-2}\rho\,\theta_2 + q^{-1}\lambda_+^{-2}U^1\theta_2,\end{aligned}$$

$$\begin{aligned}\theta_1 V^{3-} \;=\; & (1+q^{-1})\lambda_+^{-1} V^{3-}\theta_1 + (q-1)\lambda_+^{-1} V^{0-}\theta_1 \\ & - q^{-1/2}\lambda_+^{-3/2}\rho\,\theta_2,\end{aligned} \tag{150}$$
$$\theta_2 V^{3-} = q^{-1}(q^3+1)\lambda_+^{-1} V^{3-}\theta_2 - (q^2-q)\lambda_+^{-1} V^{0-}\theta_2,$$
$$\begin{aligned}\theta_1 V^{0-} \;=\; & q^{-2}(q^3+1)\lambda_+^{-1} V^{0-}\theta_1 + q^{-2}(q-1)\lambda_+^{-1} V^{3-}\theta_1 \\ & - q^{-1/2}\lambda_+^{-3/2}\rho\,\theta_2,\end{aligned} \tag{151}$$
$$\theta_2 V^{0-} = (q+1)\lambda_+^{-1} V^{0-}\theta_2 - (1-q^{-1})\lambda_+^{-1} V^{3-}\theta_2,$$

and similarly for spinors with a bar,

$$\bar\theta_1 V^{+3} = (1+q^{-1})\lambda_+^{-1} V^{+3}\bar\theta_1 + (q-1)\lambda_+^{-1} V^{+0}\bar\theta_1, \tag{152}$$
$$\begin{aligned}\bar\theta_2 V^{+3} \;=\; & q^{-1}(q^3+1)\lambda_+^{-1} V^{+3}\bar\theta_2 - (q^2-q)\lambda_+^{-1} V^{+0}\bar\theta_2 \\ & - q^{1/2}\lambda_+^{-3/2}\sigma\,\bar\theta_1,\end{aligned}$$
$$\bar\theta_1 V^{+0} = q^{-2}(q^3+1)\lambda_+^{-1} V^{+0}\bar\theta_1 + (q^{-1}-q^{-2})\lambda_+^{-1} V^{+3}\bar\theta_1, \tag{153}$$
$$\begin{aligned}\bar\theta_2 V^{+0} \;=\; & (q^{-1}-1)\lambda_+^{-1} V^{+3}\bar\theta_2 + (q+1)\lambda_+^{-1} V^{+0}\bar\theta_2 \\ & - q^{1/2}\lambda_+^{-3/2}\sigma\,\bar\theta_1,\end{aligned}$$

$$\begin{aligned}\bar{\theta}_1 V^{+-} &= q^{-3}(q^4+1+q^4\lambda_+)\lambda_+^{-2}V^{+-}\bar{\theta}_1\\&\quad+q^{-1}(q-1)^2(\lambda_++1)\lambda_+^{-2}V^{30}\bar{\theta}_1\\&\quad+(q^2\lambda_++1)q^{-5/2}\lambda\lambda_+^{-3/2}V^{+3}\bar{\theta}_2\\&\quad+q^{-1/2}(1-\lambda_+)\lambda\lambda_+^{-3/2}V^{+0}\bar{\theta}_2\\&\quad-q^{-1}\lambda\lambda_+^{-2}\sigma\,\bar{\theta}_1+q^{-2}\lambda_+^{-2}U^2\bar{\theta}_1,\end{aligned} \tag{154}$$

$$\begin{aligned}\bar{\theta}_2 V^{+-} &= q^{-3}(q^4+1+q^4\lambda_+)\lambda_+^{-2}V^{+-}\bar{\theta}_2\\&\quad+q^{-1}(q-1)^2(\lambda_++1)\lambda_+^{-2}V^{30}\bar{\theta}_2\\&\quad+q^{1/2}\lambda\lambda_+^{-3/2}(V^{0-}-V^{3-})\bar{\theta}_1\\&\quad-q^{-1}\lambda\lambda_+^{-2}\sigma\,\bar{\theta}_2-\lambda_+^{-2}U^2\bar{\theta}_2,\end{aligned}$$

$$\begin{aligned}\bar{\theta}_1 V^{30} &= q^{-1}(q^4+1+\lambda_+)\lambda_+^{-2}V^{30}\bar{\theta}_1\\&\quad+q^{-1}(q-1)^2(\lambda_++1)\lambda_+^{-2}V^{+-}\bar{\theta}_1\\&\quad+q^{-3/2}(1-\lambda_+)\lambda\lambda_+^{-3/2}V^{+3}\bar{\theta}_2\\&\quad+q^{-3/2}(q^4+\lambda_+)\lambda\lambda_+^{-3/2}V^{+0}\bar{\theta}_2\\&\quad-\lambda\lambda_+^{-2}\sigma\,\bar{\theta}_1+q^{-1}\lambda_+^{-2}U^2\bar{\theta}_1,\end{aligned} \tag{155}$$

$$\begin{aligned}\bar{\theta}_2 V^{30} &= q^{-1}(q^4+1+\lambda_+)\lambda_+^{-2}V^{30}\bar{\theta}_2\\&\quad+q^{-1}(q-1)^2(\lambda_++1)\lambda_+^{-2}V^{+-}\bar{\theta}_2\\&\quad+q^{3/2}\lambda\lambda_+^{-3/2}(V^{0-}-V^{3-})\bar{\theta}_1\\&\quad-\lambda\lambda_+^{-2}\sigma\,\bar{\theta}_2-q\lambda_+^{-2}U^2\bar{\theta}_2,\end{aligned}$$

$$\begin{aligned}\bar{\theta}_1 V^{3-} &= (q+1)\lambda_+^{-1}V^{3-}\bar{\theta}_1+(q^2-q)\lambda_+^{-1}V^{0-}\bar{\theta}_1\\&\quad+q^{1/2}\lambda\lambda_+^{-1/2}(qV^{+-}-V^{30})\bar{\theta}_2+q^{-1/2}\lambda_+^{-3/2}\sigma\,\bar{\theta}_2,\end{aligned} \tag{156}$$

$$\bar{\theta}_2 V^{3-} = q^{-2}(q^3+1)\lambda_+^{-1}V^{3-}\bar{\theta}_2-(q-1)\lambda_+^{-1}V^{0-}\bar{\theta}_2,$$

$$\begin{aligned}\bar{\theta}_1 V^{0-} &= q^{-1}(q^3+1)\lambda_+^{-1}V^{0-}\bar{\theta}_1+(1-q^{-1})\lambda_+^{-1}V^{3-}\bar{\theta}_1\\&\quad+q^{1/2}\lambda\lambda_+^{-1/2}(qV^{+-}-V^{30})\bar{\theta}_2-q^{-5/2}\lambda_+^{-3/2}\sigma\,\bar{\theta}_2,\end{aligned} \tag{157}$$

$$\bar{\theta}_2 V^{0-} = (q+q^{-1})\lambda_+^{-1}V^{0-}\bar{\theta}_2-(q^{-1}-q^{-2})\lambda_+^{-1}V^{3-}\bar{\theta}_2.$$

Next, we turn to the vector representations for the $V^{\mu\nu}$, which are seen to be

$$V^{\mu\nu}\triangleright X^\rho=(\tau^{\mu\nu})^\rho{}_\sigma X^\sigma,\quad X_\rho\triangleleft V^{\mu\nu}=X_\sigma(\tau^{\mu\nu})^\sigma{}_\rho, \tag{158}$$

with

$$(\tau^{+3})^\rho{}_\sigma=\lambda_+^{-2}\begin{pmatrix}0&-2q&-\lambda&0\\0&0&0&-2\\0&0&0&-q\lambda\\0&0&0&0\end{pmatrix}, \tag{159}$$

$$(\tau^{+0})^\rho{}_\sigma=\lambda_+^{-2}\begin{pmatrix}0&-\lambda&2q^{-1}&0\\0&0&0&-q^{-1}\lambda\\0&0&0&-(q^2+q^{-2})\\0&0&0&0\end{pmatrix},$$

$$(\tau^{+-})^{\rho}{}_{\sigma} = \lambda_+^{-2}\begin{pmatrix} 2q^{-2} & 0 & 0 & 0 \\ 0 & -2q^{-1}\lambda & -q^{-1}\lambda & 0 \\ 0 & q^{-1}\lambda & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

$$(\tau^{30})^{\rho}{}_{\sigma} = \lambda_+^{-2}\begin{pmatrix} \lambda q^{-2} & 0 & 0 & 0 \\ 0 & -q^{-1}\lambda^2 & 2q & 0 \\ 0 & q^{-1}(q^2+q^{-2}) & 0 & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix},$$

$$(\tau^{3-})^{\rho}{}_{\sigma} = \lambda_+^{-2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -q^{-1}\lambda & 0 & 0 & 0 \\ 0 & 2q^{-1} & -\lambda & 0 \end{pmatrix},$$

$$(\tau^{0-})^{\rho}{}_{\sigma} = \lambda_+^{-2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1}\lambda & 0 & 0 & 0 \\ 2q^{-2} & 0 & 0 & 0 \\ 0 & q^{-2}\lambda & -q^{-1}(q^2+q^{-2}) & 0 \end{pmatrix}.$$

Notice that rows and columns are labeled in the order $+, 3, 0, -$. For U^1, U^2, σ and ρ we obtain the matrices

$$(U^1)^{\rho}{}_{\sigma} = (U^1)^{\rho}{}_{\sigma} = -(q^2+q^{-2})\lambda_+^{-1}\delta^{\rho}{}_{\sigma}, \tag{160}$$

$$(\rho)^{\rho}{}_{\sigma} = \lambda_+^{-1}\begin{pmatrix} -q\lambda_+ & 0 & 0 & 0 \\ 0 & -2 & q\lambda & 0 \\ 0 & q^{-1}\lambda & -(q^2+q^{-2}) & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda_+ \end{pmatrix},$$

$$(\sigma)^{\rho}{}_{\sigma} = \lambda_+^{-1}\begin{pmatrix} q\lambda_+ & 0 & 0 & 0 \\ 0 & 2 & q^{-1}\lambda & 0 \\ 0 & q\lambda & (q^2+q^{-2}) & 0 \\ 0 & 0 & 0 & q^{-1}\lambda_+ \end{pmatrix},$$

and likewise for the generators of the $U_q(su_2)$-subalgebra,

$$(L^+)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & -q & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (L^-)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \end{pmatrix}, \tag{161}$$

$$(\tau^{-1/2})^{\rho}{}_{\sigma} = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}.$$

We can now proceed to write down commutation relations between the $V^{\mu\nu}$ and the components of a vector operator. Let us recall that we mean by a vector operator a set of objects X^{ρ} which transform under Lorentz transformations according to

$$[V^{\mu\nu}, X^{\rho}]_q = (\tau^{\mu\nu})^{\rho}{}_{\sigma} X^{\sigma}, \quad [X_{\sigma}, V^{\mu\nu}]_q = X_{\sigma}\,(\tau^{\mu\nu})^{\sigma}{}_{\rho}. \tag{162}$$

These relations are equivalent to those in (132) if we substitute the vector components for the general element a. By taking the coproduct of the $V^{\mu\nu}$ together with the vector representations of the Lorentz generators we can again compute from (132) the explicit form for the commutation relations between the $V^{\mu\nu}$ and the X^ρ. Unfortunately, the results are rather lengthy:

$$
\begin{aligned}
V^{+3}X^{+} &= q^{-2}(q^4+1)\lambda_+^{-1}X^{+}V^{+3} - q\lambda\lambda_+^{-1}X^{+}V^{+0},\\
V^{+3}X^{3} &= 2\lambda_+^{-1}X^{3}V^{+3} - q\lambda X^{+}V^{+-} + q\lambda\lambda_+^{-1}X^{0}V^{+3}\\
&\quad - q\lambda\lambda_+^{-1}X^{+}V^{30} + q\lambda_+^{-2}X^{+}(U^1+U^2),\\
V^{+3}X^{0} &= q^{-2}(q^4+1)\lambda_+^{-1}X^{0}V^{+3} + q^{-1}\lambda\lambda_+^{-1}X^{3}V^{+3}\\
&\quad - q^2\lambda\lambda_+^{-1}X^{+}V^{+-} + \lambda_+^{-2}X^{+}(qU^1 - q^{-1}U^2),\\
V^{+3}X^{-} &= 2\lambda_+^{-1}X^{-}V^{+3} - 2q\lambda\lambda_+^{-1}X^{3}V^{+-} + q\lambda\lambda_+^{-1}X^{0}V^{+-}\\
&\quad + q\lambda\lambda_+^{-1}X^{-}V^{+0} - q^2\lambda\lambda_+^{-1}X^{3}V^{30}\\
&\quad - q^2\lambda^2\lambda_+^{-1}X^{+}V^{0-} + q^2\lambda^2\lambda_+^{-1}X^{+}V^{3-}\\
&\quad + \lambda_+^{-2}X^{3}(U^1+U^2) - \lambda_+^{-2}X^{0}(U^1 - q^2U^2),
\end{aligned}
\tag{163}
$$

$$
\begin{aligned}
V^{+0}X^{+} &= 2\lambda_+^{-1}X^{+}V^{+0} - q^{-1}\lambda\lambda_+^{-1}X^{+}V^{+3},\\
V^{+0}X^{3} &= 2\lambda_+^{-1}X^{3}V^{+0} - \lambda\lambda_+^{-1}X^{+}V^{+3} - 2q\lambda\lambda_+^{-1}X^{+}V^{30}\\
&\quad + q\lambda\lambda_+^{-1}X^{0}V^{+0} - \lambda_+^{-2}X^{+}(q^{-1}U^1 - qU^2),\\
V^{+0}X^{0} &= q^{-2}\lambda_+^{-1}(q^4+1)X^{0}V^{+0} - q\lambda\lambda_+^{-1}X^{+}V^{30}\\
&\quad + q^{-1}\lambda\lambda_+^{-1}X^{3}V^{+0} - q^{-1}\lambda_+^{-2}X^{+}(U^1+U^2),\\
V^{+0}X^{-} &= q^{-2}\lambda_+^{-1}(q^4+1)X^{-}V^{+0} - q\lambda X^{3}V^{30} + \lambda\lambda_+^{-1}X^{0}V^{30}\\
&\quad + q^{-1}\lambda\lambda_+^{-1}X^{-}V^{+3} - q\lambda\lambda_+^{-1}X^{3}V^{+-} - q^2\lambda^2\lambda_+^{-1}X^{+}V^{0-}\\
&\quad + q^2\lambda^2\lambda_+^{-1}X^{+}V^{3-} + \lambda_+^{-2}X^{0}(q^{-2}U^1 + q^2U^2)\\
&\quad - \lambda_+^{-2}X^{3}(q^{-2}U^1 - U^2),
\end{aligned}
\tag{164}
$$

$$
\begin{aligned}
V^{+-}X^{+} &= 2\lambda_+^{-1}X^{+}V^{+-} + q^{-1}\lambda X^{3}V^{+3} - \lambda\lambda_+^{-1}X^{3}V^{+0}\\
&\quad - q^{-2}\lambda\lambda_+^{-1}X^{0}V^{+3} - q^{-2}\lambda_+^{-2}X^{+}(U^1+U^2),\\
V^{+-}X^{3} &= (2-\lambda^2)\lambda_+^{-1}X^{3}V^{+-} - \lambda X^{+}V^{3-} + q\lambda\lambda_+^{-1}X^{+}V^{0-}\\
&\quad + 2q^{-1}\lambda\lambda_+^{-1}X^{-}V^{+3} - q^{-1}\lambda\lambda_+^{-1}X^{-}V^{+0} + \lambda^2\lambda_+^{-1}X^{0}V^{+-}\\
&\quad + q^{-2}\lambda_+^{-2}X^{0}(U^1-U^2) + q^{-1}\lambda\lambda_+^{-2}X^{3}(U^1+U^2),\\
V^{+-}X^{0} &= q^{-2}(q^4+1)\lambda_+^{-1}X^{0}V^{+-} - \lambda^2\lambda_+^{-1}X^{3}V^{+-} + q^{-1}\lambda\lambda_+^{-1}X^{-}V^{+3}\\
&\quad - q^{-1}\lambda\lambda_+^{-1}X^{+}V^{3-} + \lambda_+^{-2}X^{3}(U^1 - q^{-2}U^2),\\
V^{+-}X^{-} &= 2\lambda_+^{-1}X^{-}V^{+-} - 2\lambda\lambda_+^{-1}X^{3}V^{3-} + \lambda\lambda_+^{-1}X^{3}V^{0-}\\
&\quad + \lambda\lambda_+^{-1}X^{0}V^{3-} + \lambda_+^{-2}X^{-}(U^1+U^2),
\end{aligned}
\tag{165}
$$

$$
\begin{aligned}
V^{30}X^{+} &= 2\lambda_+^{-1}X^{+}V^{30} - q^{-1}\lambda\lambda_+^{-1}X^{0}V^{+0} - q^{-1}\lambda\lambda_+^{-1}X^{3}V^{+3}\\
&\quad + 2q^{-1}\lambda\lambda_+^{-1}X^{3}V^{+0} + q^{-1}\lambda_+^{-2}X^{+}(q^{-2}U^1 - U^2),\\
V^{30}X^{3} &= (2-\lambda^2)\lambda_+^{-1}X^{3}V^{30} + q\lambda X^{+}V^{0-} + q^{-1}\lambda X^{-}V^{+0}\\
&\quad + q^{-2}\lambda\lambda_+^{-1}(1-q^3\lambda_+)X^{+}V^{3-} - q^{-2}\lambda\lambda_+^{-1}X^{-}V^{+3}\\
&\quad + \lambda^2\lambda_+^{-1}X^{0}V^{30} - \lambda\lambda_+^{-2}X^{3}(q^{-2}U^1 - U^2)\\
&\quad - \lambda_+^{-2}X^{0}(q^{-3}U^1 + qU^2),
\end{aligned}
\tag{166}
$$

$$\begin{aligned}
V^{30}X^0 &= q^{-2}(q^4+1)\lambda_+^{-1}X^0V^{30} + \lambda\lambda_+^{-1}X^+V^{0-} + \lambda\lambda_+^{-1}X^-V^{+0}\\
&\quad - q^{-1}\lambda^2\lambda_+^{-1}X^+V^{3-} - \lambda^2\lambda_+^{-1}X^3V^{30} - q^{-1}\lambda_+^{-2}X^3(U^1+U^2),\\
V^{30}X^- &= 2\lambda_+^{-1}X^-V^{30} - \lambda\lambda_+^{-1}(q+1)X^3V^{3-} - q\lambda\lambda_+^{-1}X^0V^{0-}\\
&\quad + 2q\lambda\lambda_+^{-1}X^3V^{0-} + \lambda^2\lambda_+^{-1}X^0V^{3-} - \lambda_+^{-2}X^-(q^{-1}U^1 - qU^2),
\end{aligned}$$

$$\begin{aligned}
V^{3-}X^+ &= q^{-2}(q^4+1)\lambda_+^{-1}X^+V^{3-} - q\lambda\lambda_+^{-1}X^0V^{+-}\\
&\quad - q\lambda\lambda_+^{-1}X^+V^{0-} + 2q\lambda\lambda_+^{-1}X^3V^{+-} + \lambda^2\lambda_+^{-1}X^-(V^{+3} - V^{+0})\\
&\quad - \lambda\lambda_+^{-1}X^3V^{30} - \lambda_+^{-2}X^3(U^1+U^2) - \lambda_+^{-2}X^0(q^{-2}U^1 - U^2),\\
V^{3-}X^3 &= 2\lambda_+^{-1}X^3V^{3-} + q\lambda X^-V^{+-} - q^{-1}\lambda\lambda_+^{-1}X^0V^{3-}\\
&\quad - q\lambda\lambda_+^{-1}X^-V^{30} - q^{-1}\lambda_+^{-2}X^-(U^1+U^2),\\
V^{3-}X^0 &= q^{-2}(q^4+1)\lambda_+^{-1}X^0V^{3-} - q\lambda\lambda_+^{-1}X^3V^{3-}\\
&\quad + \lambda\lambda_+^{-1}X^-V^{+-} + \lambda_+^{-2}X^-(qU^1 - q^{-1}U^2),\\
V^{3-}X^- &= 2\lambda_+^{-1}X^-V^{3-} + q\lambda\lambda_+^{-1}X^-V^{0-},
\end{aligned} \tag{167}$$

$$\begin{aligned}
V^{0-}X^+ &= 2\lambda_+^{-1}X^+V^{0-} + \lambda\lambda_+^{-1}(q+1)X^3V^{+-} - 2\lambda\lambda_+^{-1}X^3V^{30}\\
&\quad - q^{-1}\lambda\lambda_+^{-1}X^+V^{3-} + \lambda\lambda_+^{-1}X^0V^{30}\\
&\quad + \lambda^2\lambda_+^{-1}X^-(V^{+3} - V^{+0}) - \lambda^2\lambda_+^{-1}X^0V^{+-}\\
&\quad - \lambda_+^{-2}X^3(U^1 - q^{-2}U^2) - q^{-2}\lambda_+^{-2}X^0(U^1+U^2),\\
V^{0-}X^3 &= 2\lambda_+^{-1}X^3V^{0-} - \lambda X^-V^{30} - q^{-2}\lambda\lambda_+^{-1}X^-(1 - q^3\lambda_+)V^{+-}\\
&\quad - q^{-1}\lambda\lambda_+^{-1}X^0V^{0-} - q^{-1}\lambda_+^{-2}X^-(U^1 - q^{-2}U^2),\\
V^{0-}X^0 &= q^{-2}(q^4+1)\lambda_+^{-1}X^0V^{0-} - q\lambda\lambda_+^{-1}X^3V^{0-}\\
&\quad - q^{-1}\lambda\lambda_+^{-1}X^-V^{30} + q^{-1}\lambda^2\lambda_+^{-1}X^-V^{+-}\\
&\quad + \lambda_+^{-2}X^-(qU^1 + q^{-3}U^2),\\
V^{0-}X^- &= q^{-2}(q^4+1)\lambda_+^{-1}X^-V^{0-} + q^{-1}\lambda\lambda_+^{-1}X^-V^{3-}.
\end{aligned} \tag{168}$$

If we want to commute the $V^{\mu\nu}$ to the left of a vector operator, we have to apply instead

$$\begin{aligned}
X_+V^{+3} &= q^{-2}(q^4+1)\lambda_+^{-1}V^{+3}X_+ - q\lambda V^{+-}X_3\\
&\quad - q\lambda\lambda_+^{-1}V^{30}X_3 - q\lambda\lambda_+^{-1}V^{+0}X_+ - q^2\lambda\lambda_+^{-1}V^{+-}X_0\\
&\quad + q^2\lambda^2\lambda_+^{-1}(V^{3-} - V^{0-})X_- + q\lambda_+^{-2}(U^1+U^2)X_3\\
&\quad + \lambda_+^{-2}(qU^1 - q^{-1}U^2)X_0,\\
X_3V^{+3} &= 2\lambda_+^{-1}V^{+3}X_3 - 2q\lambda\lambda_+^{-1}V^{+-}X_- + q^{-1}\lambda\lambda_+^{-1}V^{+3}X_0\\
&\quad - q^2\lambda\lambda_+^{-1}V^{30}X_- + \lambda_+^{-2}(U^1+U^2)X_-,\\
X_0V^{+3} &= q^{-2}(q^4+1)\lambda_+^{-1}V^{+3}X_0 + q\lambda\lambda_+^{-1}V^{+3}X_3\\
&\quad + q\lambda\lambda_+^{-1}V^{+-}X_- - \lambda_+^{-2}(U^1 - q^2U^2)X_-,\\
X_-V^{+3} &= 2\lambda_+^{-1}V^{+3}X_- + q\lambda\lambda_+^{-1}V^{+0}X_-,
\end{aligned} \tag{169}$$

$$\begin{aligned}
X_+V^{+0} &= 2\lambda_+^{-1}V^{+0}X_+ - \lambda\lambda_+^{-1}V^{+-}X_3 - q^{-1}\lambda\lambda_+^{-1}V^{+3}X_+\\
&\quad - 2q\lambda\lambda_+^{-1}V^{30}X_3 - q\lambda\lambda_+^{-1}V^{30}X_0 + q^2\lambda^2\lambda_+^{-2}(V^{3-} - V^{0-})X_-\\
&\quad - \lambda_+^{-2}(q^{-1}U^1 - qU^2)X_3 - q^{-1}\lambda_+^{-2}(U^1+U^2)X_0,
\end{aligned} \tag{170}$$

$$
\begin{aligned}
X_3V^{+0} &= 2\lambda_+^{-1}V^{+0}X_3 - q\lambda V^{30}X_- + q^{-1}\lambda\lambda_+^{-1}V^{+0}X_0 \\
&\quad - q\lambda\lambda_+^{-1}V^{+-}X_- - \lambda_+^{-2}(q^{-2}U^1 - U^2)X_-, \\
X_0V^{+0} &= q^{-2}(q^4+1)\lambda_+^{-1}V^{+0}X_0 + \lambda\lambda_+^{-1}V^{30}X_- \\
&\quad + q\lambda\lambda_+^{-1}V^{+0}X_3 + \lambda_+^{-2}(q^{-2}U^1 + q^2U^2)X_-, \\
X_-V^{+0} &= q^{-2}(q^4+1)\lambda_+^{-1}V^{+0}X_- + q^{-1}\lambda\lambda_+^{-1}V^{+3}X_-,
\end{aligned}
$$

$$
\begin{aligned}
X_+V^{+-} &= 2\lambda_+^{-1}V^{+-}X_+ - \lambda V^{3-}X_3 - q^{-1}\lambda\lambda_+^{-1}V^{3-}X_0 \\
&\quad + q\lambda\lambda_+^{-1}V^{0-}X_3 - q^{-2}\lambda_+^{-2}(U^1 + U^2)X_+, \qquad (171)\\
X_3V^{+-} &= (2-\lambda^2)\lambda_+^{-1}V^{+-}X_3 + q^{-1}\lambda V^{+3}X_+ - \lambda\lambda_+^{-1}V^{+0}X_+ \\
&\quad + \lambda\lambda_+^{-1}V^{0-}X_- - 2\lambda\lambda_+^{-1}V^{3-}X_- - \lambda^2\lambda_+^{-1}V^{+-}X_0 \\
&\quad + q^{-1}\lambda\lambda_+^{-1}(U^1 + U^2)X_3 + \lambda_+^{-2}(U^1 - q^{-2}U^2)X_0, \\
X_0V^{+-} &= q^{-2}(q^4+1)\lambda_+^{-1}V^{+-}X_0 - q^{-2}\lambda\lambda_+^{-1}V^{+3}X_+ \\
&\quad + \lambda\lambda_+^{-1}V^{3-}X_- + \lambda^2\lambda_+^{-1}V^{+-}X_3 + \lambda_+^{-2}(q^{-2}U^1 - U^2)X_3, \\
X_-V^{+-} &= 2\lambda_+^{-1}V^{+-}X_- + 2q^{-1}\lambda\lambda_+^{-1}V^{+3}X_3 + q^{-1}\lambda\lambda_+^{-1}V^{+3}X_0 \\
&\quad - q^{-1}\lambda\lambda_+^{-1}V^{+0}X_3 + \lambda_+^{-2}(U^1 + U^2)X_-,
\end{aligned}
$$

$$
\begin{aligned}
X_+V^{30} &= 2\lambda_+^{-1}V^{30}X_+ + q^{-2}\lambda\lambda_+^{-1}(1 - q^3\lambda_+)V^{3-}X_3 \\
&\quad + q\lambda V^{0-}X_3 + \lambda\lambda_+^{-1}V^{0-}X_0 - q^{-1}\lambda^2\lambda_+^{-1}V^{3-}X_0 \\
&\quad + q^{-1}\lambda_+^{-2}(q^{-2}U^1 - U^2)X_+, \qquad (172)\\
X_3V^{30} &= (2-\lambda^2)\lambda_+^{-1}V^{30}X_3 - (q+\lambda)\lambda\lambda_+^{-1}V^{3-}X_- \\
&\quad - q^{-1}\lambda\lambda_+^{-1}V^{+3}X_+ + 2\lambda\lambda_+^{-1}(q^{-1}V^{+0} + qV^{0-})X_+ \\
&\quad - \lambda^2\lambda_+V^{30}X_0 - \lambda\lambda_+^{-1}(q^{-2}U^1 - U^2)X_3 \\
&\quad + q^{-1}\lambda_+^{-2}(U^1 - U^2)X_0, \\
X_0V^{30} &= q^{-2}(q^4+1)V^{30}X_0 - q^{-1}\lambda\lambda_+^{-1}V^{+0}X_+ - q\lambda\lambda_+^{-1}V^{0-}X_- \\
&\quad + \lambda^2\lambda_+^{-1}V^{30}X_3 + \lambda^2\lambda_+^{-1}V^{3-}X_- - \lambda_+^{-2}(q^{-3}U^1 + qU^2)X_3, \\
X_-V^{30} &= 2\lambda_+^{-1}V^{30}X_- + q^{-1}\lambda V^{+0}X_0 - q^{-2}\lambda\lambda_+^{-1}V^{+3}X_3 \\
&\quad + \lambda\lambda_+^{-1}V^{+0}X_0 - \lambda_+^{-2}(q^{-1}U^1 - qU^2)X_-,
\end{aligned}
$$

$$
\begin{aligned}
X_+V^{3-} &= q^{-2}(q^4+1)V^{3-}X_+ - q\lambda\lambda_+^{-1}V^{0-}X_+, \qquad (173)\\
X_3V^{3-} &= 2\lambda_+^{-1}V^{3-}X_3 + 2q\lambda\lambda_+^{-1}V^{+-}X_+ - \lambda\lambda_+^{-1}V^{30}X_+ \\
&\quad - q\lambda\lambda_+^{-1}V^{3-}X_0 - \lambda_+^{-2}(U^1 + U^2)X_+, \\
X_0V^{3-} &= q^{-2}(q^4+1)\lambda_+^{-1}V^{3-}X_0 - q\lambda\lambda_+^{-1}V^{+-}X_+ \\
&\quad - q^1\lambda\lambda_+^{-1}V^{3-}X_3 - \lambda_+^{-2}(q^{-2}U^1 - U^2)X_+, \\
X_-V^{3-} &= 2\lambda_+^{-1}V^{3-}X_- + q\lambda V^{+-}X_3 + \lambda\lambda_+^{-1}V^{+-}X_0 \\
&\quad + \lambda^2\lambda_+^{-1}(V^{+3} - V^{+0})X_+ + q\lambda\lambda_+^{-1}V^{0-}X_- - q\lambda\lambda_+^{-1}V^{30}X_3 \\
&\quad + \lambda_+^{-2}(qU^1 - q^{-1}U^2)X_0 - q^{-1}\lambda_+^{-2}(U^1 + U^2)X_3,
\end{aligned}
$$

$$
\begin{aligned}
X_+V^{0-} &= 2\lambda_+^{-1}V^{0-}X_+ - q^{-1}\lambda\lambda_+^{-1}V^{3-}X_+, \qquad (174)\\
X_3V^{0-} &= 2\lambda_+^{-1}V^{0-}X_3 + \lambda(q+\lambda)\lambda_+^{-1}V^{+-}X_+ - 2\lambda\lambda_+^{-1}V^{30}X_+ \\
&\quad - q\lambda\lambda_+^{-1}V^{0-}X_0 - \lambda_+^{-2}(U^1 - q^{-2}U^2)X_+,
\end{aligned}
$$

$$
\begin{aligned}
X_0 V^{0-} &= q^{-2}(q^4+1)\lambda_+^{-1} V^{0-} X_0 + \lambda\lambda_+^{-1} V^{30} X_+ \\
&\quad - q^{-1}\lambda\lambda_+^{-1} V^{0-} X_3 - \lambda^2\lambda_+^{-1} V^{+-} X_+ - q^{-2}\lambda_+^{-2}(U^1+U^2)X_+, \\
X_- V^{0-} &= q^{-2}(q^4+1)\lambda_+^{-1} V^{0-} X_- - \lambda V^{30} X_3 + q^{-1}\lambda\lambda_+^{-1} V^{3-} X_- \\
&\quad - q^{-1}\lambda\lambda_+^{-1} V^{30} X_0 + q^{-3}\lambda\lambda_+^{-1}(-q+q^{-2}\lambda_+) V^{+-} X_3 \\
&\quad - q^{-1}\lambda_+^{-2}(U^1 - q^{-1}U^2)X_3 + \lambda_+^{-2}(qU^1 + q^{-2}U^2)X_0.
\end{aligned}
$$

5.4. *Quantum Lie algebra of q-deformed Lorentz algebra and its Casimir operators*

Last but not least we would like to present the quantum Lie algebra of q-deformed Lorentz algebra. To achieve this, we calculate the adjoint action of the independent $V^{\mu\nu}$ on themselves in the fashion of the previous section and set the results equal to the q-commutators of the corresponding Lorentz generators. In the cases where the adjoint actions do not vanish, we obtain

$$
\begin{aligned}
&[V^{+3}, V^{+-}]_q = -q^2[V^{+-}, V^{+3}]_q = -\lambda_+^{-1} V^{+3}, \\
&[V^{+3}, V^{30}]_q = -q^2[V^{30}, V^{+3}]_q = -q\lambda_+^{-1} V^{+0}, \\
&[V^{+3}, V^{3-}]_q = -[V^{3-}, V^{+3}]_q = -q\lambda_+^{-1} V^{+-}, \\
&[V^{+3}, V^{0-}]_q = -[V^{0-}, V^{+3}]_q = \lambda_+^{-1}(V^{30} - \lambda V^{+-}),
\end{aligned} \tag{175}
$$

$$
\begin{aligned}
&[V^{+0}, V^{+-}]_q = -q^2[V^{+-}, V^{+0}]_q = -\lambda_+^{-1} V^{+0}, \\
&[V^{+0}, V^{30}]_q = -q^2[V^{30}, V^{+0}]_q = -\lambda_+^{-1}(q^{-1}V^{+3} + \lambda V^{+0}), \\
&[V^{+0}, V^{3-}]_q = -[V^{3-}, V^{+0}]_q = -\lambda_+^{-1} V^{30}, \\
&[V^{+0}, V^{0-}]_q = -[V^{0-}, V^{+-}]_q = q^{-1}\lambda_+^{-1} V^{+-},
\end{aligned} \tag{176}
$$

$$
\begin{aligned}
&[V^{+-}, V^{+-}]_q = -q^{-1}\lambda\lambda_+^{-1} V^{+-}, \\
&[V^{+-}, V^{30}]_q = [V^{30}, V^{+-}]_q = -q^{-1}\lambda\lambda_+^{-1} V^{30}, \\
&[V^{+-}, V^{3-}]_q = -q^2[V^{3-}, V^{+-}]_q = -\lambda_+^{-1} V^{3-}, \\
&[V^{+-}, V^{0-}]_q = -q^2[V^{0-}, V^{+-}]_q = -\lambda_+^{-1} V^{0-},
\end{aligned} \tag{177}
$$

$$
\begin{aligned}
&[V^{30}, V^{30}]_q = -q^{-1}\lambda\lambda_+^{-1}(V^{+-} + \lambda V^{30}), \\
&[V^{30}, V^{3-}]_q = -q^2[V^{3-}, V^{30}]_q = \lambda_+^{-1}(qV^{0-} - \lambda V^{3-}), \\
&[V^{30}, V^{0-}]_q = -q^2[V^{0-}, V^{30}]_q = q^{-1}\lambda_+^{-1} V^{3-}.
\end{aligned} \tag{178}
$$

Of course, the quantum Lie algebra relations are consistent with spinor and vector representation of q-deformed Lorentz algebra. This can again be checked in a familiar way by writing out q-commutators and substituting the representation matrices for the Lorentz generators.

As Casimirs of this quantum Lie algebra we have found the two operators

$$
\begin{aligned}
C^1 &= \eta_{\mu\nu}\eta_{\rho\sigma} V^{\mu\rho} V^{\nu\sigma} = 2V^{30}V^{30} + (q^2+q^{-2})V^{+-}V^{+-} \\
&\quad + 2(qV^{+0}V^{0-} - q^{-1}V^{+3}V^{3-} - q^{-3}V^{3-}V^{3+}) \\
&\quad - \lambda(V^{+3}V^{0-} + V^{+0}V^{3-} + V^{+-}V^{30} + V^{30}V^{+-}) \\
&\quad - q^{-2}\lambda(V^{3-}V^{+0} + V^{0-}V^{+3}),
\end{aligned} \tag{179}
$$

$$
\begin{aligned}
C^2 &= \varepsilon_{\mu\nu\rho\sigma} V^{\mu\nu} V^{\rho\sigma} = (3 + q^{-4} - 2q^{-3}\lambda)(V^{+3}V^{0-} - V^{+0}V^{3-}) \\
&\quad + (3 + q^{-4} - 2q^{-3})(V^{+-}V^{30} + V^{30}V^{+-}) \\
&\quad + q^{-6}(3q^4 + 1 - 2q\lambda)(V^{0-}V^{+3} - V^{3-}V^{+0}) \\
&\quad - q^{-2}\lambda(2q^2 + 2q\lambda + \lambda\lambda_+)V^{+-}V^{+-},
\end{aligned} \tag{180}
$$

with $\eta_{\mu\nu}$ and $\varepsilon_{\mu\nu\rho\sigma}$ being q-analogs of the Minkowski metric and corresponding epsilon tensor (see also Appendix A). Specifying the two Casimirs for the different representations finally yields the results:

(a) (operator representation)

$$
\begin{aligned}
C^1 &= 2q\lambda_+^{-2}(X \circ X)(\partial \circ \partial) + 2q^{-2}\lambda_+^{-2}(X \circ \partial)(X \circ \partial) \\
&\quad + 2q\lambda_+^{-1} X \circ \partial,
\end{aligned} \tag{181}
$$

$$C^2 = 0,$$

(b) (spinor representation)

$$
\begin{aligned}
C^1 &= [[3]]_{q^{-4}}\lambda_+^{-2}\mathbb{1}_{2\times 2}, \\
C^2 &= [[3]]_{q^4}(3 + q^4 + 2q^3\lambda)\mathbb{1}_{2\times 2},
\end{aligned} \tag{182}
$$

(c) (vector representation)

$$
\begin{aligned}
C^1 &= 2[[3]]_{q^{-4}}\lambda_+^{-2}\mathbb{1}_{3\times 3}, \\
C^2 &= 0.
\end{aligned} \tag{183}
$$

6. Conclusion

Let us end with a few comments on our results. We dealt with quantum algebras describing q-deformed versions of physical symmetries. In this way we considered $U_q(su_2)$, $U_q(so_4)$, and q-deformed Lorentz algebra. It was our aim to provide a consistent framework of basic definitions and relations which allow for representation theoretic investigations in physics.

In each case the starting point of our reasonings was the realization of symmetry generators within q-deformed differential calculus. We listed the relations of the corresponding symmetry algebras and presented expressions for coproducts and antipodes on symmetry generators. We realized that the Hopf structure of the symmetry generators allows us to define q-analogs of classical commutators. Furthermore, we concerned ourselves with q-deformed versions of such finite dimensional representations as play an important role in physics, i.e. spinor and vector representations. With the help of these representations we were able to write down q-deformed commutation relations between symmetry generators and components of a spinor or vector operator. Moreover, we calculated the adjoint action of the symmetry generators on each other and obtained relations for quantum Lie algebras in this way. Finally, we gave expressions for the corresponding Casimir operators and specified them for the different representations.

Our reasonings were in complete analogy to the classical situation, but compared with the undeformed case our results are modified by terms proportional to $\lambda = q - q^{-1}$. Hence, we regain in the classical limit as $q \rightarrow 1$ the familiar expressions. This observation nourishes the hope that a field theory based on quantum group symmetries can be developed along the same lines as its undeformed counterpart.

In this sense we also hope that implementing our results on a computer algebra system like Mathematica will prove useful in a systematic search for new q-identities. Furthermore, such an undertaking can be helpful for making it obvious that everything presented in this article works fine.

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Appendix A. q-Deformed quantum spaces

The aim of this appendix is the following. For quantum spaces of physical importance we list their defining commutation relations as well as the nonvanishing elements of their quantum metric and q-deformed epsilon tensor.

The coordinates θ^α, $\alpha = 1, 2$, of the two-dimensional antisymmetrized quantum plane fulfill the relation [33,41]

$$\theta^1\theta^2 = -q^{-1}\theta^2\theta^1, \quad q \in \mathbb{R}, q > 1, \tag{A.1}$$

whereas the spinor metric is given by a matrix $\varepsilon^{\alpha\beta}$ with nonvanishing elements

$$\varepsilon^{12} = q^{-1/2}, \quad \varepsilon^{21} = -q^{1/2}. \tag{A.2}$$

Furthermore, we can raise and lower indices as usual, i.e.

$$\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta, \quad \theta_\alpha = \varepsilon_{\alpha\beta}\theta^\beta, \tag{A.3}$$

where $\varepsilon_{\alpha\beta}$ denotes the inverse of $\varepsilon^{\alpha\beta}$.

In the case of three-dimensional q-deformed Euclidean space the commutation relations between its coordinates X^A, $A \in \{+, 3, -\}$, read

$$\begin{aligned} X^3X^\pm &= q^{\pm 2}X^\pm X^3, \\ X^-X^+ &= X^+X^- + \lambda X^3X^3. \end{aligned} \tag{A.4}$$

The nonvanishing elements of the quantum metric are

$$g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -q^{-1}. \tag{A.5}$$

Now, the covariant coordinates are given by

$$X_A = g_{AB}X^B, \tag{A.6}$$

with g_{AB} being the inverse of g^{AB}. The nonvanishing components of the three-dimensional q-deformed epsilon tensor take the form

$$\begin{aligned} &\varepsilon_{-3+} = 1, \quad \varepsilon_{3-+} = -q^{-2}, \\ &\varepsilon_{-+3} = -q^{-2}, \quad \varepsilon_{+-3} = q^{-2}, \end{aligned} \tag{A.7}$$

$$\varepsilon_{3+-} = q^{-2}, \quad \varepsilon_{+3-} = -q^{-4},$$
$$\varepsilon_{333} = -q^{-2}\lambda.$$

In Section 3 we need especially

$$\epsilon_{AB}{}^{C} \equiv g^{CD}\varepsilon_{ABD}, \quad \epsilon^{AB}{}_{C} \equiv g^{AD}g^{BE}\varepsilon_{DEC}. \tag{A.8}$$

Next we come to four-dimensional Euclidean space. For its coordinates X^i, $i = 1, \ldots, 4$, we have the relations

$$\begin{aligned} &X^1X^j = qX^jX^1, \quad j = 1, 2,\\ &X^jX^4 = qX^4X^j,\\ &X^2X^3 = X^3X^2,\\ &X^4X^1 = X^1X^4 + \lambda X^2X^3. \end{aligned} \tag{A.9}$$

The metric has the nonvanishing components

$$g^{14} = q^{-1}, \quad g^{23} = g^{32} = 1, \quad g^{41} = q. \tag{A.10}$$

Its inverse denoted by g_{ij} can again be used to introduce covariant coordinates, i.e.

$$X_i = g_{ij}X^j. \tag{A.11}$$

For the epsilon tensor of q-deformed Euclidean space with four dimensions one can find as components

$$\begin{aligned} &\varepsilon_{1234} = q^4, \quad \varepsilon_{1432} = -q^2, \quad \varepsilon_{2413} = -q^2,\\ &\varepsilon_{2134} = -q^3, \quad \varepsilon_{4132} = q^2, \quad \varepsilon_{4213} = q,\\ &\varepsilon_{1324} = -q^4, \quad \varepsilon_{3412} = q^2, \quad \varepsilon_{2341} = -q^2,\\ &\varepsilon_{3124} = q^3, \quad \varepsilon_{4312} = -q, \quad \varepsilon_{3241} = q^2,\\ &\varepsilon_{2314} = q^2, \quad \varepsilon_{1243} = -q^3, \quad \varepsilon_{2431} = q,\\ &\varepsilon_{3214} = -q^2, \quad \varepsilon_{2143} = q^2, \quad \varepsilon_{4231} = -1,\\ &\varepsilon_{1342} = q^3, \quad \varepsilon_{1423} = q^2, \quad \varepsilon_{3421} = -q,\\ &\varepsilon_{3142} = -q^2, \quad \varepsilon_{4123} = -q^2, \quad \varepsilon_{4321} = 1. \end{aligned} \tag{A.12}$$

In addition to this there are the non-classical components

$$\varepsilon_{3232} = -\varepsilon_{2323} = q^2\lambda. \tag{A.13}$$

Now, we come to q-deformed Minkowski space [29,37,42] (for other deformations of spacetime and their related symmetries we refer the reader to [8,10,12,13,23,27]). Its coordinates are subject to the relations

$$\begin{aligned} &X^\mu X^0 = X^0X^\mu, \quad \mu \in \{0, +, -, 3\},\\ &X^3X^\pm - q^{\pm 2}X^\pm X^3 = -q\lambda X^0X^\pm,\\ &X^-X^+ - X^+X^- = \lambda(X^3X^3 - X^0X^3), \end{aligned} \tag{A.14}$$

and its metric is given by

$$\eta^{00} = -1, \quad \eta^{33} = 1, \quad \eta^{+-} = -q, \quad \eta^{-+} = -q^{-1}. \tag{A.15}$$

As usual, the metric can be used to raise and lower indices. In analogy to the classical case the q-deformed epsilon tensor has as nonvanishing components

$$\begin{aligned}
&\varepsilon_{+30-} = -q^{-4}, \quad \varepsilon_{+-03} = q^{-2}, \quad \varepsilon_{3-+0} = q^{-2},\\
&\varepsilon_{3+0-} = q^{-2}, \quad \varepsilon_{-+03} = -q^{-2}, \quad \varepsilon_{-3+0} = -1,\\
&\varepsilon_{+03-} = q^{-4}, \quad \varepsilon_{0-+3} = -q^{-2}, \quad \varepsilon_{30-+} = q^{-2},\\
&\varepsilon_{0+3-} = -q^{-4}, \quad \varepsilon_{-0+3} = q^{-2}, \quad \varepsilon_{03-+} = -q^{-2},\\
&\varepsilon_{30+-} = -q^{-2}, \quad \varepsilon_{+3-0} = q^{-4}, \quad \varepsilon_{3-0+} = -q^{-2},\\
&\varepsilon_{03+-} = q^{-2}, \quad \varepsilon_{3+-0} = -q^{-2}, \quad \varepsilon_{-30+} = 1,\\
&\varepsilon_{+0-3} = -q^{-2}, \quad \varepsilon_{+-30} = -q^{-2}, \quad \varepsilon_{0-3+} = 1,\\
&\varepsilon_{0+-3} = q^{-2}, \quad \varepsilon_{-+30} = q^{-2}, \quad \varepsilon_{-03+} = -1,
\end{aligned} \tag{A.16}$$

and

$$\begin{aligned}
&\varepsilon_{0-0+} = q^{-1}\lambda, \quad \varepsilon_{-0+0} = -q^{-1}\lambda,\\
&\varepsilon_{0333} = -q^{-2}\lambda, \quad \varepsilon_{3330} = q^{-2}\lambda,\\
&\varepsilon_{3033} = +q^{-2}\lambda, \quad \varepsilon_{3030} = -q^{-2}\lambda,\\
&\varepsilon_{3303} = -q^{-2}\lambda, \quad \varepsilon_{+0-0} = -q^{-3}\lambda,\\
&\varepsilon_{0303} = q^{-2}\lambda, \quad \varepsilon_{0+0-} = q^{-3}\lambda.
\end{aligned} \tag{A.17}$$

References

[1] C. Bauer, H. Wachter, Operator representations on quantum spaces, Eur. Phys. J. C 31 (2003) 261–275. math-ph/0201023.

[2] L.C. Biedenharn, M.A. Lohe, Quantum Group Symmetry and q-Tensor algebras, World Scientific, Singapore, 1995.

[3] L.C. Biedenharn, M. Tartini, On q-tensor operators for quantum groups, Lett. Math. Phys. 20 (1990) 271–278.

[4] C. Blohmann, Spin representations of the q-Poincaré algebra, Ph.D. Thesis, Ludwig-Maximilians-Universität München, Fakultät für Physik (2001).

[5] C. Blohmann, Free q-deformed relativistic wave equations by representation theory, Eur. Phys. J. C 30 (2003) 435–445. hep-th/0111172.

[6] U. Carow-Watamura, M. Schlieker, M. Scholl, S. Watamura, Tensor representations of the quantum group $SL_q(2)$ and quantum Minkowski space, Z. Phys. C 48 (1990) 159–165.

[7] U. Carow-Watamura, M. Schlieker, S. Watamura, $SO_q(N)$-covariant differential calculus on quantum space and deformation of Schrödinger equation, Z. Phys. C 49 (1991) 439–446.

[8] L. Castellani, Differential calculus on $ISO_q(N)$, Quantum Poincaré Algebra and q-gravity (preprint). hep-th/9312179.

[9] B.L. Cerchiai, J. Wess, q-deformed Minkowski space based on a q-Lorentz algebra, Eur. Phys. J. C 5 (1998) 553–566. math.QA/9801104.

[10] M. Chaichian, A.P. Demichev, Quantum Poincaré group without dilatation and twisted classical algebra, J. Math. Phys. 36 (1995) 398–413.

[11] M. Chaichian, A.P. Demichev, Introduction to Quantum Groups, World Scientific, Singapore, 1996.

[12] M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, On a Lorentz-Invariant interpretation of noncommutative space-time and its implications on noncommutative QFT, Phys. Lett. B 604 (2004) 98–102. hep-th/0408062.

[13] V.K. Dobrev, New q-Minkowski space-time and q-Maxwell equations hierarchy from q-conformal invariance, Phys. Lett. B 341 (1994) 133–138.

[14] V.G. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation, Soviet Math. Dokl. 32 (1985) 254–258.

[15] V.G. Drinfeld, Quantum groups, in: A.M. Gleason (Ed.), Proceedings of the International Congress of Mathematicians, Amer. Math. Soc., 1986, pp. 798–820.

[16] M. Fichtmüller, A. Lorek, J. Wess, q-deformed phase space and its lattice structure, Z. Phys. C 71 (1996) 533–538.

[17] G. Fiore, Quantum groups $SO_q(N)$, $Sp_q(N)$ have q-determinants, too, J. Phys. A 27 (1994) 3795–3802.
[18] G. Fiore, Realization of $U_q(so_N)$ within the differential algebra on $R_q(N)$, Comm. Math. Phys. 169 (1995) 475–500. hep-th/9403023.
[19] H. Grosse, C. Klimčik, P. Prešnajder, Towards finite quantum field theory in non-commutative geometry, Internat. J. Theoret. Phys. 35 (1996) 231–244. hep-th/9505175.
[20] W. Heisenberg, Über die in der Theorie der Elementarteilchen auftretende universelle Länge, Ann. Physics 32 (1938) 20–33.
[21] M. Jimbo, A q-analogue of $U(g)$ and the Yang–Baxter equation, Lett. Math. Phys. 10 (1985) 63–69.
[22] A. Klimyk, K. Schmüdgen, Quantum Groups and their Representations, Springer Verlag, Berlin, 1997.
[23] F. Koch, E. Tsouchnika, Construction of θ-Poincaré Algebras and their invariants on $\mathcal{M}_\theta$, Nuclear Phys. B 717 (2005) 387–403. hep-th/0409012.
[24] P.P. Kulish, N.Y. Reshetikhin, Quantum linear problem for the sine–Gordon equation and higher representations, J. Sov. Math. 23 (1983) 24–35.
[25] A. Lorek, W.B. Schmidke, J. Wess, $SU_q(2)$-covariant $\hat{R}$-Matrices for Reducible Representations, Lett. Math. Phys. 31 (1994) 279–288.
[26] A. Lorek, W. Weich, J. Wess, Non-commutative Euclidean and Minkowski structures, Z. Phys. C 76 (1997) 375–386. q-alg/9702025.
[27] J. Lukierski, A. Nowicki, H. Ruegg, New quantum Poincare algebra and κ-deformed field theory, Phys. Lett. B 293 (1992) 344–352.
[28] S. Majid, On the q-regularisation, Internat. J. Modern Phys. A 5 (1990) 4689–4696.
[29] S. Majid, Examples of braided groups and braided matrices, J. Math. Phys. 32 (1991) 3246–3253.
[30] S. Majid, Algebras and Hopf algebras in braided categories, Lec. Notes Pure Appl. Math. 158 (1994) 55–105.
[31] S. Majid, Quantum and braided Lie algebras, J. Geom. Phys. 13 (1994) 307–356.
[32] S. Majid, Foundations of Quantum Group Theory, University Press, Cambridge, 1995.
[33] Y.J. Manin, Quantum groups and Non-Commutative Geometry, Centre de Recherche Mathematiques, Montreal, 1988.
[34] D. Mikulovic, A. Schmidt, H. Wachter, Grassmann variables on quantum spaces (preprint). hep-th/0407273.
[35] H. Ocampo, $SO_q(4)$ quantum mechanics, Z. Phys. C 70 (1996) 525–530.
[36] R. Oeckl, Braided quantum field theory, Comm. Math. Phys. 217 (2001) 451–473.
[37] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, q-deformed Poincaré algebra, Comm. Math. Phys. 150 (1992) 495–518.
[38] P. Podles, S.L. Woronowicz, Quantum deformation of Lorentz group, Comm. Math. Phys. 130 (1990) 381–431.
[39] N.Yu. Reshetikhin, L.A. Takhtadzhyan, L.D. Faddeev, Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1990) 193–225.
[40] M. Rohregger, J. Wess, q-deformed Lorentz algebra in Minkowski phase space, Eur. Phys. J. C 7 (1999) 177–183.
[41] M. Schlieker, W. Scholl, Spinor calculus for quantum groups, Z. Phys. C 47 (1990) 625–628.
[42] W.B. Schmidke, J. Wess, B. Zumino, A q-deformed Lorentz algebra in Minkowski phase space, Z. Phys. C 52 (1991) 471–476.
[43] A. Schmidt, H. Wachter, Superanalysis on quantum spaces (preprint). hep-th/0411180.
[44] P. Schupp, P. Watts, B. Zumino, Bicovariant quantum algebras and quantum Lie algebras, Comm. Math. Phys. 157 (1993) 305–329.
[45] H.S. Snyder, Quantized space-time, Phys. Rev. 71 (1947) 38–41.
[46] X.C. Song, Covariant differential calculus on quantum Minkowski space and q-analog of Dirac equation, Z. Phys. C 55 (1992) 417–422.
[47] A. Sudbery, Quantum differential calculus and Lie algebras, Internat. J. Modern Phys. A (1993) 228–231.
[48] H. Wachter, M. Wohlgenannt, $*$-Products on quantum spaces, Eur. Phys. J. C 23 (2002) 761–767. hep-th/0103120.
[49] H. Wachter, q-Integration on quantum spaces, Eur. Phys. J. C 32 (2004) 281–297. hep-th/0206083.
[50] H. Wachter, q-Exponentials on quantum spaces, Eur. Phys. J. C 37 (2004) 379–389. hep-th/0401113.
[51] H. Wachter, q-Translations on quantum spaces (preprint). hep-th/0410205.
[52] J. Wess, B. Zumino, Covariant differential calculus on the quantum hyperplane, Nucl. Phys. B. Suppl. 18 (1991) 302–312.
[53] S. Wolfram, The Mathematica Book, 4th edition, University Press, Cambridge, 1999.
[54] S.L. Woronowicz, Compact matrix pseudo groups, Comm. Math. Phys. 111 (1987) 613–665.